LECTURES 21-22

Dates of Lectures: March 29 and 31, 2022

We fix a ring R throughout these lectures. All modules appearing are R-modules, unless otherwise specified.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

Much of the notes here are re-cycled from older courses of mine, and so may talk about objects, subobjects etc, i.e. the underlying context is that of an abelian category. In the older courses, $\mathscr{A}$ was an arbitrary abelian category, and I didn't have the energy to change the notations or the references to objects and subobjects. In practical terms, set $\mathrm{Mod}_R = \mathscr{A}$, and you are good to go. There will be some bewildering statements (I did make some noises about the existence of countable direct sums, but if $\mathscr{A} = \mathrm{Mod}_R$, that is always assured).

I have combined Lectures 21 and 22.

1. Double complexes

1.1. **Standard Double Complexes.** A *double complex* in $\mathscr{A}$, or sometimes in our class, a *standard double complex* in $\mathscr{A}$, consists of data $A^{\bullet\bullet} = (A, \partial_1, \partial_2)$, where

$$A = (A^{p,q})_{(p,q)\in\mathbf{Z}\times\mathbf{Z}}$$

is a family of objects in $\mathscr{A}$, and

$$\partial_1 = (\partial_1^{p,q})_{(p,q)\in\mathbf{Z}} \qquad \partial_2 = (\partial_2^{p,q})_{(p,q)\in\mathbf{Z}}$$

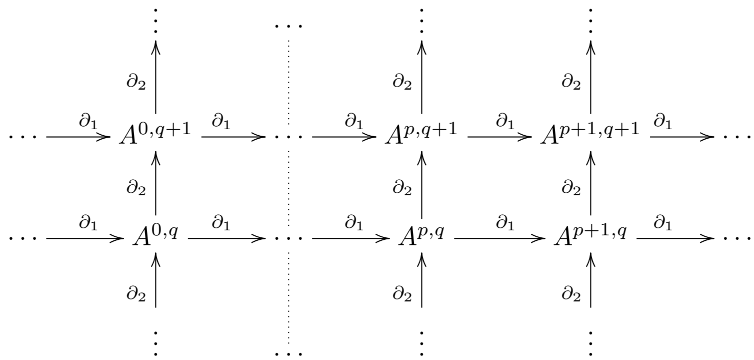
are two families of morphisms

$$\partial_1^{p,q}\colon A^{p,q} \to A^{p+1,q} \qquad \partial_2^{p,q}\colon A^{p,q} \to A^{p,q+1}$$

such that

$$\partial_1\partial_1 = 0 \qquad \partial_2\partial_2 = 0 \qquad \partial_1\partial_2 = \partial_2\partial_1.$$

We often suppress the superscripts p, q when these are either immaterial or easily deducible from the context. Thus, e.g., we write ∂_2 for $\partial_2^{p,q}$. The maps ∂_1 and ∂_2 will be called partial coboundaries, and when we wish to be more specific, they will be called *horizontal* and *vertical* (partial) coboundaries respectively. The data fits into a commutative diagram, whose rows and columns are complexes.



Next consider the direct sum[1]

$$\mathrm{Tot}^n A^{\bullet\bullet} := \bigoplus_{p+q=n} A^{p,q}.$$

Define

$$\partial^n \colon \mathrm{Tot}^n A^{\bullet\bullet} \to \mathrm{Tot}^{n+1} A^{\bullet\bullet}$$

by the formula

$$\partial^n = \sum_{p+q=n} \{\partial_1^{p,q} + (-1)^p \partial_2^{p,q}\}.$$

The map within "curly brackets" can be regarded as a map $A^{p,q} \to \mathrm{Tot}^{n+1} A^{\bullet\bullet}$, taking values in the subobject $A^{p+1,q} \oplus A^{p,q+1}$ of $\mathrm{Tot}^{n+1} A^{\bullet\bullet}$, whence by the definition of direct sum, the map ∂^n makes sense.

Evidently

$$\partial^{n+1} \circ \partial^n = 0$$

for every $n \in \mathbf{Z}$ by the relations given between ∂_1 and ∂_2. We have therefore a complex $(\mathrm{Tot}^\bullet A^{\bullet\bullet}, \partial)$, called the *total complex* associated to the double complex $A^{\bullet\bullet}$.

A morphism of double complexes $f \colon A^{\bullet\bullet} \to B^{\bullet\bullet}$ is (of course) a family of maps $f^{p,q} \colon A^{p,q} \to B^{p,q}$, one for each ordered pair of integers (p, q), which commute with vertical and horizontal coboundaries. This naturally induces a map of complexes $\mathrm{Tot}\, f \colon \mathrm{Tot}^\bullet A^{\bullet\bullet} \to \mathrm{Tot}^\bullet B^{\bullet\bullet}$

1.2. **Anti-commutative double complexes.** In much of the pre-Grothendieck literature, double complexes mean a variant of our standard double complexes. The only difference is that the grids in the diagram on the last page anti-commute rather than commute. In greater detail, for this course, data of the form $K^{\bullet\bullet} = (K, d_1, d_2)$ represents an *anti-commuting* double complex if K is a family $(K^{p,q})$ of objects in $\mathscr{A}$ indexed by $\mathbf{Z} \times \mathbf{Z}$ and $d_1 = (d_1^{p,q} \colon K^{p,q} \to K^{p+1,q})$, $d_2 = (d_2^{p,q} \colon K^{p,q} \to K^{p,q+1})$ are families of maps indexed by $(p, q) \in \mathbf{Z} \times \mathbf{Z}$, called the *horizontal* and *vertical* partial coboundaries respectively, such that

$$d_1 d_1 = 0 \qquad d_2 d_2 = 0, \qquad d_1 d_2 = -d_2 d_1.$$

[1]This is where our assumption that $\mathscr{A}$ has countable direct sums comes into play. Alternately, one can assume that the displayed direct sum for $\mathrm{Tot}^n A^{\bullet\bullet}$ is finite for every $n \in \mathbf{Z}$.

We set (and please pay attention to the notation, especially the accent on the top left)

$$ {}'\mathrm{Tot}^n K^{\bullet\bullet} := \bigoplus_{p+q=n} K^{p,q} $$

and define

$$ d^n \colon {}'\mathrm{Tot}^n K^{\bullet\bullet} \to {}'\mathrm{Tot}^{n+1} K^{\bullet\bullet} $$

by the formula

$$ d^n = \sum_{p+q=n} (d_1^{p,q} + d_2^{p,q}) $$

without any sign of the form $(-1)^p$ intervening. It is easy to see, with $d := (d^n)_{n\in\mathbf{Z}}$, that $({}'\mathrm{Tot}^\bullet K^{\bullet\bullet}, d)$ is a complex. We call this complex the total complex associated with the anti-commuting double complex $K^{\bullet\bullet}$.

I will leave the task of defining maps of anti-commuting double complexes to you.

1.3. **Bounded double complexes.** Let $C^{\bullet\bullet}$ be a double complex (standard or anti-commutative). We say it is *bounded on the left* if there is an integer p_0 such that

$$ C^{p,q} = 0, \qquad p < p_0. $$

If this happens we sometimes say $C^{\bullet\bullet}$ is *bounded on the left by* p_0. Similarly $C^{\bullet\bullet}$ is *bounded below (by* q_0*)* if there exists an integer q_0 such that

$$ C^{p,q} = 0 \qquad q < q_0. $$

I leave to you the fun task of defining terms like *bounded on the right* and *bounded above.*

Note that if $C^{\bullet\bullet}$ is bounded on the left and below (resp. above and to the right) it lives in a translate of the first quadrant (resp. third quadrant) and as such the direct sum

$$ \bigoplus_{p+q=n} C^{p,q} $$

is actually a finite sum[2] for each n. So in such instances, one can define $\mathrm{Tot}^n C^{\bullet\bullet}$ or ${}'\mathrm{Tot}^n C^{\bullet\bullet}$ (as the case may be) without insisting that $\mathscr{A}$ have countable direct sums. In fact we will largely be dealing with such situations.

1.4. **The transpose of $A^{\bullet\bullet}$.** There is an obvious notion of a transpose of a double complex. Suppose $(A^{\bullet\bullet}, \partial_1, \partial_2)$ is a double complex. One can define a new double complex $(X^{\bullet\bullet}, \partial_h, \partial_v)$, with

$$ X^{pq} = A^{qp}, \qquad \partial_h^{pq} = \partial_2^{qp}, \qquad \text{and} \qquad \partial_v^{pq} = \partial_1^{qp}. $$

It is easy to see that there is a natural a isomorphism of complexes

$$ \theta \colon \mathrm{Tot}^\bullet(A^{\bullet\bullet}) \xrightarrow{\sim} \mathrm{Tot}^\bullet(X^{\bullet\bullet}) \tag{1.4.1} $$

[2]Draw a picture with such quadrant translates, and look at the intersection of such quadrant translates with lines having slope -1.

given at the n^{th} level by the sum of the maps $\theta^{p,q}\colon A^{pq} \to X^{qp} = A^{pq}$, $p+q=n$, where $\theta^{p,q}$ is multiplication by $(-1)^{pq}$. Indeed, if $n = p+q$,

$$\begin{aligned}\partial_X^{q,p} \circ \theta^n = \partial_X^{q,p} \circ \theta^{p,q} = (-1)^{pq}(\partial_h + (-1)^q \partial_v) &= (-1)^{pq}((-1)^q \partial_1 + \partial_2) \\ &= (-1)^{pq+q}\partial_1 + (-1)^{pq+p}(-1)^p\partial_2 \\ &= \theta^{p+1,q}\partial_1 + \theta^{p,q+1}(-1)^p\partial_2 \\ &= \theta^{n+1} \circ \partial_A^{p,q}.\end{aligned}$$

which means θ is a cochain map, and therefore necessarily an isomorphism of complexes, since it is so in each graded piece. The "matrix" of θ^n, regarded as a map from the direct sum $A^{0,n} \oplus A^{1,n-1} \oplus \cdots \oplus A^{n,0}$, to the direct sum $X^{0,n} \oplus X^{1,n-1} \oplus \cdots \oplus X^{n,0}$, is of the form

$$\theta^n = \begin{bmatrix} 0 & \dots & 0 & \theta^{0,n} \\ 0 & \dots & \theta^{1,n-1} & 0 \\ \vdots & \iddots & \vdots & \vdots \\ \theta^{n,0} & \dots & 0 & 0 \end{bmatrix}$$

1.5. **Notations.** Let $A^{\bullet\bullet}$ be a double complex, and m an integer. Then $A^{\bullet\bullet}_{m\geqslant 0}$ will mean the double complex (with obvious vertical and horizontal differentials) whose $(p,q)^{\text{th}}$ graded piece is

(1.5.1) $$A^{pq}_{m\geqslant 0} = \begin{cases} A^{pq} & \text{if } p \geq m \\ 0 & \text{otherwise.} \end{cases}$$

Similarly one can define $A^{\bullet\bullet}_{\leqslant m}$, and set $A^{\bullet\bullet}_{>m} = A^{\bullet\bullet}_{\geqslant m+1}$, and $A^{\bullet\bullet}_{<m} = A^{\bullet\bullet}_{\leqslant m-1}$. These are the column truncations of $A^{\bullet\bullet}$. Similarly there are row truncations ${}_{\leqslant m}A^{\bullet\bullet}$, ${}_{<m}A^{\bullet\bullet}$, ${}_{\geqslant m}A^{\bullet\bullet}$, and ${}_{>m}A^{\bullet\bullet}$.

It is important to remember that while $A^{\bullet\bullet}_{\geqslant m}$ is a sub-double complex of $A^{\bullet\bullet}$, in general $A^{\bullet\bullet}_{\leqslant m}$ needn't be one. On the other hand $A^{\bullet\bullet}_{\leqslant m}$ is a quotient double complex of $A^{\bullet\bullet}$, but $A^{\bullet\bullet}_{\geqslant m}$, in general, need not be one.

For every $m \in \mathbf{Z}$ we have a short exact sequence of double complexes

(1.5.2) $$0 \longrightarrow A^{\bullet\bullet}_{\geqslant m} \longrightarrow A^{\bullet\bullet} \longrightarrow A^{\bullet\bullet}_{<m} \longrightarrow 0.$$

Theorem 1.5.3. *Let $A^{\bullet\bullet}$ be a double complex whose columns (respectively rows) are exact. Then* $\mathrm{Tot}^\bullet(A^{\bullet\bullet})$ *if one of the following holds:*

(a) *$A^{\bullet\bullet}$ is bounded to the right and to the left (respectively, above and below);*
(b) *$A^{\bullet\bullet}$ is bounded below and to the left;*
(c) *$A^{\bullet\bullet}$ is bounded above and to the right.*

Proof. By using the transpose of $A^{\bullet\bullet}$ and the isomorphism (1.4.1) if necessary, we only need to consider the case where the columns of $A^{\bullet\bullet}$ are exact.

Let us assume (a). We prove (b). Without loss of generality, we may assume (by translating if necessary), that $A^{\bullet\bullet}$ lives in the first quadrant, i.e. $A^{pq} = 0$ if p or q is negative. In this case, clearly $\mathrm{H}^n(\mathrm{Tot}^\bullet(A^{\bullet\bullet})) = \mathrm{H}^n(\mathrm{Tot}^\bullet(A^{\bullet\bullet}_{\leqslant n+1}))$. By (a), $\mathrm{H}^n(\mathrm{Tot}^\bullet(A^{\bullet\bullet}_{\leqslant n+1})) = 0$ since $A^{\bullet\bullet}_{\leqslant n+1}$ is bounded to the left and to the right. This proves (b). Next we prove (c), assuming (a). This time we assume, without loss of generality, that $A^{\bullet\bullet}$ lives in the third quadrant. In this case, clearly $\mathrm{H}^n(\mathrm{Tot}^\bullet(A^{\bullet\bullet})) = \mathrm{H}^n(\mathrm{Tot}^\bullet(A^{\bullet\bullet}_{\geqslant n-1}))$, and $A^{\bullet\bullet}_{\geqslant n-1}$ is bounded to the right and to the left.

It remains to prove (a). The assertion is obvious if all but one column of $A^{\bullet\bullet}$ is zero. Assume without loss of generality, that $A^{\bullet\bullet}$ is bounded on the left by zero, i.e. $A^{\bullet\bullet} = A^{\bullet\bullet}_{\geqslant 0}$. Since $A^{\bullet\bullet}$ is also bounded on the right, there exists $n \geq 0$ such that $A^{\bullet\bullet} = A^{\bullet\bullet}_{\leqslant n}$. By way induction, we assume that the assertion is true when $n = m-1$ for some $m > 0$. Now suppose $n = m$. Then $A^{\bullet\bullet}_{\geqslant m}$ has all its columns equal to zero except (possibly) its m^{th} column. The exact sequence (1.5.2) gives us an exact sequence of complexes

$$(\#) \qquad 0 \longrightarrow \mathrm{Tot}^{\bullet}(A^{\bullet\bullet}_{\geqslant m}) \longrightarrow \mathrm{Tot}^{\bullet}(A^{\bullet\bullet}) \longrightarrow \mathrm{Tot}^{\bullet}(A^{\bullet\bullet}_{<m}) \longrightarrow 0.$$

Since $A^{\bullet\bullet}_{\geqslant m}$ is has all except (possibly) one column equal to zero, $\mathrm{Tot}^{\bullet}(A^{\bullet\bullet}_{\geqslant m})$ is exact. By our induction hypothesis, $\mathrm{Tot}^{\bullet}(A^{\bullet\bullet}_{<m})$ is exact. It follows from the long exact sequence associated to (#) that $\mathrm{Tot}^{\bullet}(A^{\bullet\bullet})$ is exact. □

1.5.4. Terminology. Before stating the next result, some terminology is in order. We say that the double complex $A^{\bullet\bullet}$ is *bounded on the left by* m if $A^{pq} = 0$ for $p < m$, i.e. if $A^{dubull} = A^{\bullet\bullet}_{\geqslant m}$. In this case we say m is a *left bound* of $A^{\bullet\bullet}$. $A^{\bullet\bullet}$ is *bounded to the right by* m if $A^{\bullet\bullet} = A^{\bullet\bullet}_{\leqslant m}$. If $A^{\bullet\bullet}$ is bounded on the right by m, then m is called a *right bound of* $A^{\bullet\bullet}$. Similarly one can define what it means for $A^{\bullet\bullet}$ to be bounded above by m or bounded below by m. An *upper bound*, (respectively, a *lower bound*) of bounded above (respectively bounded below) double complex can be defined in an obvious way.

The double complex $A^{\bullet\bullet}$ is said to be *concentrated* in the m^{th} column (respectively m^{th} row) if $A^{pq} = 0$ for $p \neq m$ (respectively $q \neq m$). Note that in this happens if and only if $A^{\bullet\bullet}$ is bounded on the left and right (repectively above and below) by m.

Theorem 1.5.5. *Let $A^{\bullet\bullet}$ be double complex which is bounded to the left by m and bounded below by n.*

(a) *Let $Z^{\bullet\bullet}$ be a double complex concentrated in the m^{th} column, $\varphi\colon Z^{\bullet\bullet} \to A^{\bullet\bullet}$ a map of double complexes, such that for each $q \in \mathbf{Z}$ the sequence*

$$0 \longrightarrow Z^{m,q} \xrightarrow{\varphi^{mq}} A^{mq} \xrightarrow{\partial_1} A^{m+1,q} \xrightarrow{\partial_1} \dots \xrightarrow{\partial_1} A^{m+k,q} \xrightarrow{\partial_1} \dots$$

is exact. Then $\mathrm{Tot}(\varphi)\colon \mathrm{Tot}^{\bullet}(Z^{\bullet\bullet}) \to \mathrm{Tot}^{\bullet}(A^{\bullet\bullet})$ is a quasi-isomorphism.

(b) *Let $Z^{\bullet\bullet}$ be a double complex concentrated in the n^{th} row, $\varphi\colon Z^{\bullet\bullet} \to A^{\bullet\bullet}$ a map of double complexes, such that, for each $p \in \mathbf{Z}$ the sequence*

$$0 \longrightarrow Z^{pn} \xrightarrow{\varphi^{p,n}} A^{pn} \xrightarrow{\partial_2} A^{p,n+1} \xrightarrow{\partial_2} \dots \xrightarrow{\partial_2}\xrightarrow{\partial_2} A^{p,n+k} \xrightarrow{\partial_2} \dots$$

is exact. Then $\mathrm{Tot}(\varphi)\colon \mathrm{Tot}^{\bullet}(Z^{\bullet\bullet}) \to \mathrm{Tot}^{\bullet}(A^{\bullet\bullet})$ is a quasi-isomorphism.

Proof. Clearly (b) follows from (a) by taking transposes. It remains to prove (a). Without loss of generality, assume $m = 0$ and $n = 0$ so that $A^{\bullet\bullet}$ is a first quadrant double complex. Let $\widetilde{A}^{\bullet\bullet}$ be the double complex obtained by setting

$$\widetilde{A}^{pq} = \begin{cases} 0 & \text{if } p < -1 \\ Z^{0,q} & \text{if } p = -1 \\ A^{pq} & \text{if } p \neq -1 \end{cases}$$

with the horizontal and vertical differentials that of $A^{\bullet\bullet}$ in the first quadrant, and with the vertical differential on the $(-1)^{\text{th}}$ column being that of the 0^{th} column of $Z^{\bullet\bullet}$. The horizontal differential on the $(-1)^{\text{th}}$ column is given by φ. It is straightforward to see that $\widetilde{A}^{\bullet\bullet}$ is indeed a double complex. By Theorem 1.5.3, $\mathrm{Tot}^\bullet(\widetilde{A}^{\bullet\bullet})$ is exact. Moreover, if $\psi = \mathrm{Tot}(\varphi)$, then it is easy to check that

$$\mathrm{Tot}^\bullet(\widetilde{A}^{\bullet\bullet}) = C_\psi^\bullet$$

where $C_\psi^\bullet$ is the mapping cone of ψ. Since $C_\psi^\bullet$ is exact, $\psi = \mathrm{Tot}(\varphi)$ is a quasi isomorphism, as asserted. $\square$

1.5.6. Analogous statements can be made for complexes bounded to the right and above (translates of third quadrant double complexes). I leave the formulation and proof to you. An easy proof is obtained by working with $\mathscr{A}^\circ$, the opposite category of $\mathscr{A}$. Then the result just falls out.

2. **The homotopy category**

2.1. **A basic result.** The following result is extremely useful

Proposition 2.1.1. *Let $C^\bullet$ be bounded below exact sequence of R-modules, $E^\bullet$ a complex of injectives modules, and $\phi\colon C^\bullet \to E^\bullet$ a map of complexes. Then $\phi \sim 0$.*

Proof. By translating $C^\bullet$ and $E^\bullet$ if necessary, we may assume $C^p = 0$ for $p < 0$. We have a commutative diagram of complexes with the top row exact:

$$\begin{array}{ccccccccccccc}
 & & 0 & \longrightarrow & C^0 & \xrightarrow{d_C^0} & C^1 & \xrightarrow{d_C^1} & \cdots\cdots & \xrightarrow{d_C^{n-2}} & C^{n-1} & \xrightarrow{d_C^{n-1}} & C^n & \xrightarrow{d_C^n} & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle \phi^0} & & \downarrow{\scriptstyle \phi^1} & & & & \downarrow{\scriptstyle \phi^{n-1}} & & \downarrow{\scriptstyle \phi^n} & & \\
 & \longrightarrow & E^{-1} & \xrightarrow[d_E^1]{} & E^0 & \xrightarrow[d_E^0]{} & E^1 & \xrightarrow[d_E^1]{} & \cdots\cdots & \xrightarrow[d_E^{n-2}]{} & E^{n-1} & \xrightarrow[d_E^{n-1}]{} & E^n & \xrightarrow[d_E^n]{} & \cdots
\end{array}$$

Set $k^i = 0$ for $i \le 0$. Since the top row is exact, $C^0 \to C^1$ is an injective map. Since E^0 is an injective object, we get a map $k^1\colon C^1 \to E^0$ such that $k^1 \circ d_C^0 = \phi^0$. Let n be a positive integer. Suppose k^i have been defined for $i \le n-1$ so that the homotopy condition is satisfied up to the $(n-2)^{\text{th}}$ stage. We have

$$d_E^{n-2} \circ (\phi^{n-2} - (k^{n-1} \circ d_C^{n-2})) = d_E^{n-2} \circ (d_E^{n-3} \circ k^{n-2}) = (d_E^{n-2} \circ d_E^{n-3}) \circ k^{n-2} = 0$$

whence

$$d_E^{n-2} k^{n-1} d_C^{n-2} = d_E^{n-2} \phi^{n-2} = \phi^{n-1} d_C^{n-2}.$$

The image of d_C^{n-2} is $B^{n-1}(C^\bullet) = Z^{n-1}(C^\bullet)$, the latter equality due to the fact that $C^\bullet$ is exact. Thus the map

$$C^{n-2} \xrightarrow{d_C^{n-2}} Z^{n-1}(C^\bullet)$$

is an epimorphism. Therefore, using the fact that $d_E^{n-2} k^{n-1} d_C^{n-2} = \phi^{n-1} d_C^{n-2}$, we get that $d_E^{n-2} k^{n-1}|_{Z^{n-1}(C^\bullet)} = \phi^{n-1}|_{Z^{n-1}(C^\bullet)}$. In other words

$$(\phi^{n-1} - d_E^{n-2} k^{n-1})|_{Z^{n-1}(C^\bullet)} = 0.$$

It follows that there is a map $\kappa\colon C^{n-1}/Z^{n-1}(C^\bullet) \to E^{n-1}$ such that the following diagram commutes:

$$\begin{array}{ccc}
C^{n-1} & \twoheadrightarrow & C^{n-1}/Z^{n-1}(C^\bullet) \\
{\scriptstyle \phi^{n-1} - d_E^{n-2} k^{n-1}}\downarrow & \swarrow{\scriptstyle \kappa} & \\
E^{n-1} & &
\end{array}$$

Now $C^{n-1}/Z^{n-1}(C^\bullet) = B^n(C^\bullet) = Z^n(C^\bullet)$ is a subobject of C^n. Since E^{n-1} is injective, therefore κ can be "extended" from $Z^n(C^\bullet)$ to C^n giving us a map $k^n\colon C^n \to E^{n-1}$. By the construction of k^n we have $k^n d_C^{n-1} + d_E^{n-2} k^{n-1} = \phi^{n-1}$. In other words we have defined k^n so that the homotopy condition for ϕ extends to the $(n-1)^{\text{th}}$ stage. This completes the proof. ☐

2.2. **The category $\mathbf{K}(R)$.** As before, let $\mathbf{C}(R)$ be the category of complexes of of R-modules (morphisms being maps of complexes), and let $\mathbf{K}(R)$ be the category whose objects are the same as the objects of $\mathbf{C}(R)$, but whose morphisms are homotopy equivalence classes of maps in $\mathbf{C}(R)$. The notion of a quasi-isomorphism continues to make sense in $\mathbf{K}(R)$ in view of Problem 8 of Homework 5. $\mathbf{K}^+(R)$ will denote the full subcategory of $\mathbf{K}(R)$ consisting of complexes which are bounded below.

2.2.1. The embedding of Mod_R into $\mathbf{C}(R)$ and $\mathbf{K}(R)$ First note that Mod_R embeds into $\mathbf{C}(R)$ where we are using the notations of §2.2. Indeed if $M \in \mathrm{Mod}_R$, then M can be identified with the complex which is M at the 0^{th} spot and zero elsewhere. It is clear that if $M, N \in \mathrm{Mod}_R$ then a map $\alpha\colon M \to N$ in $\mathbf{C}(R)$ is exactly the same as map in Mod_R (check!). Moreover, since M and N when regarded as complexes are zero in non-zero degrees, α is the only morphism in its homotopy class, since any map k^n from the n^{th} spot of M to the $(n-1)^{\text{st}}$ spot of N must be the zero map since one of the source or the target of k^n is zero. Thus R embeds into $\mathbf{K}(R)$ too. Sometimes, when we wish to think of an A-module M as a complex, we write $M[0]$.

Note that if $B^\bullet$ is a complex with $B^n = 0$ for $n < 0$, and $A \in \mathrm{Mod}_R$, then a map $\alpha\colon A \to B^\bullet$ in $\mathbf{K}(R)$ corresponds to a unique map $A \to B^\bullet$ in $\mathbf{C}(R)$, which we also denote α. To give such a map amounts to giving a map $A \to B^0$, which we again denote by α, such that we get a complex

$$0 \longrightarrow A \xrightarrow{\alpha} B^0 \longrightarrow B^1 \longrightarrow \dots \longrightarrow B^n \longrightarrow \dots \tag{2.2.2}$$

with A sitting on the $(-1)^{\text{th}}$ spot.

Here is a reinterpretation of Proposition 2.1.1

Proposition 2.2.3. *Let $C^\bullet$ and $E^\bullet$ be bounded below complexes, with $C^\bullet$ exact and $E^\bullet$ a complex of injectives. Then*

$$\mathrm{H}^n\Big(\mathrm{Hom}^\bullet_R(C^\bullet, E^\bullet)\Big) = 0, \qquad n \in \mathbf{Z}.$$

Proof. We have

$$\begin{aligned}\mathrm{H}^n\Big(\mathrm{Hom}^\bullet_R(C^\bullet, E^\bullet)\Big) &= \mathrm{H}^0\Big(\mathrm{Hom}^\bullet_R(C^\bullet, E^\bullet)[n]\Big)\\ &= \mathrm{H}^0\Big(\mathrm{Hom}^\bullet_R(C^\bullet, E^\bullet[n])\Big).\end{aligned}$$

By Problem 8 of Homework 5, $\mathrm{H}^0(\mathrm{Hom}^\bullet_R(C^\bullet, E^\bullet[n]))$ is the group of cochain maps from $C^\bullet$ to $E^\bullet[n]$ modulo the subgroup of cochain maps which are homotopic to zero. Now $C^\bullet$ is bounded below and exact and $E^\bullet[n]$ is an injective complex. It follows by Proposition 2.1.1 that $\mathrm{H}^0(\mathrm{Hom}^\bullet_R(C^\bullet, E^\bullet[n])) = 0$. ☐

2.3. **Resolutions.** Let $A \in \mathrm{Mod}_R$. A (classical) *resolution* of A by a complex $B^\bullet$ with $B^n = 0$ for $n < 0$ is an exact sequence

$$0 \longrightarrow A \xrightarrow{\varepsilon} B^0 \longrightarrow B^1 \longrightarrow B^1 \longrightarrow \ldots \longrightarrow B^n \longrightarrow \ldots \tag{2.3.1}$$

where the arrows (other than ε) are the coboundaries in $B^\bullet$. This is equivalent to giving a map$A \xrightarrow{\varepsilon} B^\bullet$ in $\mathbf{C}(R)$ or $\mathbf{K}(R)$ which is a quasi-isomorphism. Classically, the map ε is called the *augumentation map*. Nowadays we often simply think of it as a quasi-isomorphism. Often, in modern terminology, we drop the requirement that $B^\bullet$ be concentrated in non-negative degrees and simply call any quasi-isomorphism $A \to B^\bullet$ a resolution of A. In fact, we sometimes disregard the requirement of A being an object of R and call any quasi-isomorphism $A^\bullet \to B^\bullet$ (with $A^\bullet$ in $\mathbf{C}(R)$ or $\mathbf{K}(R)$), a resolution of $A^\bullet$ by $B^\bullet$).

2.3.2. If we use homology complexes (chain complexes), then resolutions of $A \in \mathrm{Mod}_R$ are (classically) exact sequences of the form

$$\ldots \longrightarrow B_n \longrightarrow \ldots B_1 \longrightarrow B_0 \xrightarrow{\varepsilon} A \to 0 \tag{2.3.2.1}$$

where $B_\bullet$ is a homology complex concentrated in non-negative degrees. All the remarks and observations made about resolutions involving cohomology complexes (cochain complexes) apply in this situation too, with suitable modifications.

One immediate consequence of Proposition 2.2.3 (the re-interpretation of Proposition 2.1.1) is,

Proposition 2.3.3. *Consider a diagram in the* $\mathbf{K}(R)$

$$\begin{array}{ccc} A^\bullet & & \\ {\scriptstyle\varphi}\downarrow & \searrow^{\alpha} & \\ B^\bullet & \dashrightarrow & E^\bullet \end{array}$$

where $A^\bullet$ and $B^\bullet$ are bounded below, φ is a quasi isomorphism, and $E^\bullet$ a complex of injectives. Then there exists a unique map $\beta\colon B^\bullet \to E^\bullet$ in $\mathbf{K}(R)$ filling the broken arrow above in a manner such that the resulting diagram commutes in $\mathbf{K}(R)$.

Proof. Take any representative $f\colon A^\bullet \to B^\bullet$ in $\mathbf{C}(R)$ of $\varphi \in \mathbf{K}(R)$. Let $C^\bullet$ be the mapping cone of f. Consider the associated exact sequence of complexes

$$0 \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow A^\bullet[1] \longrightarrow 0 \tag{†}$$

associated to the map f. Since φ (and hence f) is a quasi-isomorphism, $C^\bullet$ is exact, whence by Proposition 2.2.3 we have $\mathrm{H}^n(\mathrm{Hom}^\bullet_{\mathscr{A}}(C^\bullet, E^\bullet)) = 0$ for all $n \in \mathbf{Z}$. The exact sequence (†) is *semi-split*, i.e. for each $n \in \mathbf{Z}$, the exact sequence

$$0 \longrightarrow B^n \longrightarrow C^n \longrightarrow A^{n+1} \longrightarrow 0 \tag{$\dagger_n$}$$

arising from (†), is split. Since $(\dagger)_n$ is split, for every $M \in \mathrm{Mod}_R$, the sequence arising from taking "transposes"

$$0 \longrightarrow \mathrm{Hom}_R(A^{n+1}, M) \longrightarrow \mathrm{Hom}_R(C^n,\, M) \longrightarrow \mathrm{Hom}_R(B^n,\, M) \longrightarrow 0$$

also splits, since $\mathrm{Hom}_R(-,\, M)$ respects direct sums. In particular, the above sequence is exact. From this it follows that we have an exact sequence of complexes

$$0 \longrightarrow \mathrm{Hom}^\bullet(A^\bullet[1], E^\bullet) \longrightarrow \mathrm{Hom}^\bullet(C^\bullet, E^\bullet) \longrightarrow \mathrm{Hom}^\bullet(B^\bullet, E^\bullet) \longrightarrow 0.$$

Since $\mathrm{H}^n(\mathrm{Hom}^\bullet_{\mathcal{A}}(C^\bullet, E^\bullet)) = 0$ for every integer n, the long exact sequence of groups induced by the short exact sequence of complexes above gives us isomorphisms

$$\mathrm{H}^n\Big(\mathrm{Hom}^\bullet(B^\bullet, E^\bullet)\Big) \xrightarrow[\text{via } \varphi]{\sim} \mathrm{H}^n\Big(\mathrm{Hom}^\bullet(A^\bullet, E^\bullet)\Big), \qquad n \in \mathbf{Z}.$$

Setting $n = 0$ in the above family of isomorphisms, we get the asserted result. In greater detail, the element $\alpha \in \mathrm{H}^0(\mathrm{Hom}^\bullet(B^\bullet, E^\bullet))$ gives us an element on the right side of the isomorphism

$$\mathrm{H}^0\Big(\mathrm{Hom}^\bullet(B^\bullet, E^\bullet)\Big) \xrightarrow[\text{via } \varphi]{\sim} \mathrm{H}^0(\mathrm{Hom}^\bullet(A^\bullet, E^\bullet)).$$

There is a unique element β on the left side corresponding to it. This β fills the dotted arrow in the statement of the Proposition and is the unique one which does so. $\square$