LECTURES 19 AND 20

Dates of Lectures: March 22 and 24, 2022

We fix a ring R throughout these lectures. All modules appearing are R-modules, unless otherwise specified. Complexes will be complexes of R-modules, unless otherwise specified. I have combined Lectures 19 and 20, and changed the order of presentation of the topics.

1. Split exact sequences

1.1. **Standard split exact sequences.** A *standard split exact sequence* is a sequence of R-modules of the form

$$0 \longrightarrow M \xrightarrow{\ i\ } M \oplus N \xrightarrow{\ \pi\ } N \longrightarrow 0 \tag{1.1.1}$$

where i is the canonical inclusion and π the canonical surjection. In other words $i = \left[\begin{smallmatrix}\mathbf{1}_M\\0\end{smallmatrix}\right]$ and $\pi = [0\ \mathbf{1}_N]$ in the notation we introduced in § 1.1 of Lecture 18.

A *split exact sequence* is a sequence

$$0 \longrightarrow M \xrightarrow{\ \alpha\ } C \xrightarrow{\ \beta\ } N \longrightarrow 0 \tag{1.1.2}$$

which is isomorphic to a standard split exact sequence. In other words we have a commutative diagram with the middle downward arrow an isomorphism:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{\ \alpha\ } & C & \xrightarrow{\ \beta\ } & C & \longrightarrow & 0 \\
 & & \| & & \downarrow\wr & & \| & & \\
0 & \longrightarrow & M & \xrightarrow[\ i\ ]{} & M \oplus N & \xrightarrow[\ \pi\ ]{} & N & \longrightarrow & 0
\end{array}$$

The following result is elementary.

Proposition 1.1.3. *Let*

$$0 \longrightarrow M \xrightarrow{\ \alpha\ } C \xrightarrow{\ \beta\ } N \longrightarrow 0$$

be a short exact sequence. The following are equivalent.

(a) *The sequence is split exact.*
(b) *There exists an R-map $\tau\colon C \to M$ such that $\tau \circ \alpha = \mathbf{1}_M$.*
(c) *There exists an R-map $\sigma\colon N \to C$ such that $\beta \circ \sigma = \mathbf{1}_N$.*

Proof. Omitted, since the proof is straightforward. If you wish to have some fun, see if you can prove it without chasing elements, using only the universal properties of kernels and cokernels, so that your proof now works on additive and abelian categories (whatever these are). The proof without elements is also easy, but the novelty might be interesting. □

2. **Functorial properties of** $\mathrm{Hom}_R^\bullet$

2.1. Let $\varphi\colon A^\bullet \to B^\bullet$ be a map of complexes. Let $C^\bullet$ be a third complex. For each $n \in \mathbf{Z}$ we have a map

$$(*) \qquad \mathrm{Hom}_R^n(C^\bullet, A^\bullet) \xrightarrow{\text{via } \varphi} \mathrm{Hom}_R^n(C^\bullet, B^\bullet)$$

given by $(f^j)_{j\in\mathbf{Z}} \mapsto (\varphi^j \circ f^j)_{j\in\mathbf{Z}}$. Similarly, we have maps, one for each integer n,

$$(**) \qquad \mathrm{Hom}_R^n(B^\bullet, C^\bullet) \xrightarrow{\text{via } \varphi} \mathrm{Hom}_R^n(A^\bullet, C^\bullet)$$

given by $(g^j)_{j\in\mathbf{Z}} \mapsto (g^j \circ \varphi^j)_{j\in\mathbf{Z}}$. As n varies one checks that the maps $(*)$ and $(**)$ give maps of complexes, namely:

$$(2.1.1) \qquad \mathrm{Hom}_R^\bullet(C^\bullet, \varphi)\colon \mathrm{Hom}_R^\bullet(C^\bullet, A^\bullet) \longrightarrow \mathrm{Hom}_R^\bullet(C^\bullet, B^\bullet)$$

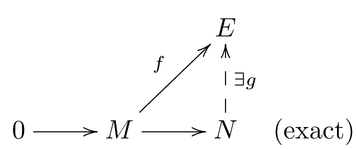
and

$$(2.1.2) \qquad \mathrm{Hom}_R^\bullet(\varphi, C^\bullet)\colon \mathrm{Hom}_R^\bullet(B^\bullet, C^\bullet) \longrightarrow \mathrm{Hom}_R^\bullet(A^\bullet, C^\bullet).$$

The verification that $(*)$ and $(**)$ give rise to maps of complexes is left to you as is the verification that $\mathrm{Hom}_R^\bullet(C^\bullet, \psi\circ\varphi) = \mathrm{Hom}_R^\bullet(C^\bullet, \psi)\circ\mathrm{Hom}_R^\bullet(C^\bullet, \varphi)$ and $\mathrm{Hom}_R^\bullet(\psi\circ\varphi, C^\bullet) = \mathrm{Hom}_R^\bullet(\varphi, C^\bullet)\circ\mathrm{Hom}_R^\bullet(\psi, C^\bullet)$ for a second map ψ such that $\psi\circ\varphi$ is meaningful.

3. **Injective Modules**

3.1. **Another definition of injective modules.** For $E \in \mathrm{Mod}_R$ we know that $h_E = \mathrm{Hom}_R(-, E)$ is left exact. For h_E to be exact it is therefore sufficient (and clearly necessary) to check that $\mathrm{Hom}_R(N, E) \to \mathrm{Hom}_R(M, E)$ is surjective whenever $0 \to M \to N$ is an exact sequence. In other words E is an injective module if and only if for every diagram of R-maps of the kind below, the broken arrow can be filled to make the diagram commute.[1]



This is often given as the definition of an injective module. We will use this characterisation of injectivity in what follows.

3.2. **Baer's criterion.** Here is an extremely useful result.

Theorem 3.2.1. (Baer's Criterion) *A module E is injective if and only if every R-map $\phi\colon I \to E$ from an ideal I of R extends to an R-map $\psi\colon R \to E$, i.e. if and only if the natural map* $\mathrm{Hom}_R(R, E) \to \mathrm{Hom}_R(I, E)$ *(given by "restriction to I") is surjective for every ideal I of R.*

Proof. If E is injective, clearly E-valued R-maps from ideals of R can always be extended to E-valued R-maps on R.

Suppose E-valued R-maps from ideals of R can always be extended to E-valued R-maps on R. We wish to show that E is injective. To that end, suppose M is a submodule of a module N and $f\colon M \to E$ is an R-map. We have to prove that there exists an R-map $g\colon N \to E$ such that $g|_M = f$. Let Σ be the set of pairs

[1]The phrase "to make the diagram commute" is often omitted, and I will do so in what follows.

(M', f'), where M' is a submodule of N containing M, and $f'\colon M \to N$ an R-map extending f. Since $(M, f) \in \Sigma$, $\Sigma \neq \emptyset$. Define a partial order $\prec$ on Σ in the obvious way: $(M', f') \prec (M'', f'')$ if $M' \subset M''$ and $f' = f''|_{M'}$. If $\{(M_\lambda, f_\lambda)\}_\lambda$ is a totally ordered chain in Σ, then $M^* = \cup_\lambda M_\lambda$ is a submodule of N (the total order is needed for this). Moreover, if $x \in M'$, then $x \in M_\lambda$ for some λ, and we can define $f'(x)$ to be $f_\lambda(x)$. If x also lies M_{λ_1}, then by the total order on our chain, either $M_\lambda \subset M_{\lambda_1}$ or $M_{\lambda_1} \subset M_\lambda$. In either case $f_{\lambda_1}(x) = f_\lambda(x)$. We therefore have a well defined map $f'\colon M' \to E$, and $f|_{M_\lambda} = f_\lambda$. Thus $(M', f') \in \Sigma$. By Zorn's Lemma, Σ has a maximal element $(\widetilde{M}, \widetilde{f})$.

We have to show that $\widetilde{M} = N$. Suppose it is not. Then there exists an element $x \in N \smallsetminus \widetilde{M}$. Let

$$I = \{r \in R \mid rx \in \widetilde{M}\}.$$

I is evidently an ideal of R. Define

$$\phi\colon I \longrightarrow E$$

by the rule $\phi(r) = \widetilde{f}(rx)$ for $r \in I$. It is easy to see that ϕ is an R-map. By our hypotheses on E, we have an R-map $\psi\colon R \to E$ which restricts to ϕ on I. Let $T = \langle \widetilde{M}, x\rangle$. Then $\widetilde{M} \subset T$ and $\widetilde{M} \neq T$.

Let $h\colon T \to E$ be defined by the rule $h(\widetilde{m} + rx) = \widetilde{f}(\widetilde{m}) + \psi(r)$, where $\widetilde{m} \in \widetilde{M}$ and $r \in R$. We need to check that this is well defined. Suppose $\widetilde{m}_1, \widetilde{m}_2 \in \widetilde{M}$, $r_1, r_2 \in R$ are such that $\widetilde{m}_1 + r_1x = \widetilde{m}_2 + r_2x$. Then $(r_2 - r_1)x = \widetilde{m}_1 - \widetilde{m}_2 \in \widetilde{M}$. This means $r_2 - r_1 \in I$, whence $\psi(r_2 - r_1) = \phi(r_2 - r_1) = \widetilde{f}((r_2 - r_1)x)$. In other words $\psi(r_1) + \widetilde{f}(\widetilde{m}_1) = \psi(r_2) + \widetilde{f}(\widetilde{m}_2)$. Thus h is well defined. It is clear that h is an R-map. Now h is clearly an extension of $\widetilde{f}$. Hence (T, h) is in Σ, contradicting the maximality of $(\widetilde{M}, \widetilde{f})$ in Σ. Thus $\widetilde{M} = N$ and we are done. □

3.3. **Direct sums and direct product of injectives.** Injectives are well behaved with respect to direct sums and finite direct sums. In the Noetherian case, they are well behaved with respect to arbitrary direct sums.

Lemma 3.3.1. *A direct summand of an injective module is injective.*

Proof. Let E be injective, and suppose E' is a direct summand of E. Then $E = E' \oplus E''$. Let $i\colon E' \hookrightarrow E$ be the natural inclusion and $\pi\colon E \to E'$ the obvious projection. Suppose M is a submodule of a module N and we have an R-map $f\colon M \to E'$. Since E is injective, the map $i \circ f\colon M \to E$ extends to an R map $h\colon N \to E$. Let $g\colon N \to E'$ be given by $g = \pi \circ h$. Then g is an extension of f. Thus E' is injective. □

Lemma 3.3.2. *An arbitrary direct product of injective modules is injective.*

Proof. Suppose we have a diagram of R-modules (this means all arrows are R-maps) with the row being exact,

$$\begin{array}{ccccc} & & & & \prod_{\lambda\in\Lambda} E_\lambda \\ & & & \overset{f}{\nearrow} & \uparrow \exists g \\ 0 & \longrightarrow & M & \longrightarrow & N \end{array}$$

with each E_λ, $\lambda \in \Lambda$, injective. We would like to fill the broken arrow. Let $E = \prod_{\lambda\in\Lambda} E_\lambda$, and for $\lambda \in \Lambda$, let $\pi_\lambda\colon E \to E_\lambda$ be the canonical projection. Then

$f = (f_\lambda)$, where $f_\lambda = \pi_\lambda \circ f$, $\lambda \in \Lambda$. Each f_λ extends to an R-maps $g_\lambda \colon N \to E_\lambda$, since E_λ is injective. Set $g = (g_\lambda)$. It is clear that the broken arrow in the diagram can be filled by g. □

Corollary 3.3.3. *A finite direct sum of injective modules is injective.*

Proof. A finite direct sum is the same as a finite direct product. □

Remark 3.3.4. It is not true in general that the arbitrary direct sum of injectives is injective. However, if the underlying ring R is Noetherian this is true, and that is what we prove next.

Proposition 3.3.5. *Let R be Noetherian. Then an arbitrary direct sum of injectives is injective.*

Proof. Suppose we have a diagram of R-maps with the E_λ, $\lambda \in \Lambda$ injective, I an ideal of R, and the row the natural exact sequence (i.e. the arrow $I \to R$ is the natural inclusion).

$$\begin{array}{ccccc} & & & & \bigoplus_{\lambda\in\Lambda} E_\lambda \\ & & & \overset{\phi}{\nearrow} & \uparrow \exists\psi \\ 0 & \longrightarrow & I & \longrightarrow & R \end{array}$$

According to Baer's Criterion, it is enough for us to fill the broken arrow. Since R is Noetherian, I is finitely generated. It follows that $\phi(I)$ is a finitely generated submodule of $\bigoplus_{\lambda\in\Lambda} E_\lambda$. Therefore there exist a finite number of indices $\lambda_1, \dots, \lambda_d$ in Λ such that $\phi(I) \subset \bigoplus_{i=1}^{d} E_{\lambda_i} = E'$ (say). Then ϕ factrors as

$$I \xrightarrow{\phi'} E' \subset \bigoplus_{\lambda\in\Lambda} E.$$

Since E' is injective, being a finite direct sum of injectives, we have and R-map $g' \colon R \to E'$ extending ϕ'. Define $g \colon R \to E$ as the composite $R \xrightarrow{g'} E' \subset \bigoplus_{\lambda\in\Lambda} E$. Clearly g fills the broken arrow in the diagram above. □

3.4. **Divisible modules.** . An element of $r \in R$ is a *non zero dvisior of R* if the "multiplication by r" map $R \xrightarrow{r} R$ is injective. We often just write "r is an NZD" or "r is an NZD of R" instead of the longer form above. $\mathrm{NZD}(R)$ denotes the set of NZDs in R.

A module M is said to be *divisible* if for every non zero divisor r of R and every element $m \in M$, we can find $m' \in M$ such that $m = rm'$. In other words M is divisible if for every NZD r of R, the map $M \xrightarrow{r} M$ is surjective.

Lemma 3.4.1. *An injective module is divisible.*

Proof. Let $r \in \mathrm{NZD}(R)$. We have an exact sequence

$$0 \longrightarrow R \xrightarrow{r} R$$

Let us apply the exact functor $\mathrm{Hom}_R(-, E)$ to the above sequence. We get the exact sequence

$$\begin{array}{ccccc} \mathrm{Hom}_R(R,E) & \xrightarrow{r} & \mathrm{Hom}_R(R,E) & \longrightarrow & 0 \\ \| & & \| & & \\ E & \xrightarrow[r]{} & E & \longrightarrow & 0 \end{array}$$

Thus $E \xrightarrow{r} E$ is surjective. ☐

The converse need not be true. However, if R is a PID, we have the following result.

Proposition 3.4.2. *Let R be a PID, and M a divisible R-module. Then M is injective.*

Proof. Since R is a PID, it is Noetherian, and hence we can apply Baer's criterion. Accordingly, suppose I is an ideal of R and $\phi\colon I \to M$ an R-map. We have to extend ϕ to R. If $I = 0$ there is nothing to prove. We assume $I \neq 0$. Since R is a PID, $I = \langle x \rangle$ for some non zero element $x \in R$. Let $m = \phi(x)$. Since R is an integral domain, x is a NZD of R. Now M is divisible, and hence there exists $m' \in M$ such that $m = xm'$. Define

$$\psi\colon R \to M$$

by the rule $\psi(r) = rm'$, $r \in R$. For $r \in R$ we have $\psi(rx) = rxm' = rm = r\phi(x) = \phi(rx)$, and hence ψ extends ϕ. Moreover, it is clearly an R-map. Thus M is injective. ☐

Examples 3.4.3. One can generate a number of examples of injective modules using Proposition 3.4.2.

1. Let R be a PID and $K = \mathbf{Q}(R)$ its quotient field, i.e. its field of fractions. Then K is a divisible R-module (by definition of the field of fractions!) and hence it is injective.

2. Let R be PID and E an injective module. Then every quotient E/M of E by a submodule M of E, is injective. Indeed, E being injective, is divisible. If $0 \neq r \in R$, and $[x] = x + M$ is an element of E/M (with $x \in E$), then we can find $y \in E$ such that $x = ry$, and it is immediate that $[x] = r[y]$. Thus E/M is divisible, and since R is a PID, it is injective.

3. A ring R is said to be *Gorenstein* if it has a finite injective resolution, i.e. there is a *finite length* exact sequence

$$0 \longrightarrow R \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \ldots \longrightarrow E^{n-1} \longrightarrow E^n \longrightarrow 0$$

with each E^i injective. Such rings are special and important in commutative algebra and in algebraic geometry. Combining the above two examples, if R is a PID and K its fraction field, then

$$0 \longrightarrow R \longrightarrow K \longrightarrow K/R \longrightarrow 0$$

is an injective resolution of R. Thus every PID is Gorenstein.

4. From 1, $\mathbf{Q}$ is an injective $\mathbf{Z}$-module. From 3, so is $\mathbf{Q}/\mathbf{Z}$.

3.5. Mod_R **has enough injectives.** Here is the first step towards the theory of derived functors in homological algebra.

Lemma 3.5.1. *Let M be a $\mathbf{Z}$-module. Then M can be embedded into an injective $\mathbf{Z}$-module.*

Proof. Let m_λ, $\lambda \in \Lambda$, be a set of generators for M. There exists a surjective $\mathbf{Z}$-map

$$\bigoplus_{\lambda \in \Lambda} \mathbf{Z} \longrightarrow\!\!\!\!\rightarrow M$$

given by $e_\lambda \mapsto m_\lambda$, where $\{e_\lambda\}$ is the canonical basis for the free $\mathbf{Z}$-module $\bigoplus_{\lambda\in\Lambda}\mathbf{Z}$. Let K be the kernel of this map. Then

$$M \cong \frac{\bigoplus_{\lambda\in\Lambda}\mathbf{Z}}{K} \subset \frac{\bigoplus_{\lambda\in\Lambda}\mathbf{Q}}{K}.$$

From 4 of 3.4.3, we know that $\mathbf{Q}$ is an injective $\mathbf{Z}$-module. Since $\mathbf{Z}$ is Noetherian, $\bigoplus_{\lambda\in\Lambda}\mathbf{Q}$ is an injective $\mathbf{Z}$-module. From 3 of 3.4.3, $\bigoplus_{\lambda\in\Lambda}\mathbf{Q}/K$ is an injective $\mathbf{Z}$-module. □

3.5.2. The above statement is usually phrased as "the category $\mathrm{Mod}_{\mathbf{Z}}$ has *enough injectives*". Enough to define and use right derived functors, that is. More generally an abelian category $\mathscr{A}$ (whatever that is) is said to have enough injectives if every object of $\mathscr{A}$ can be embedded in an injective object. Sheaves on topological spaces have enough injectives, which is why, from a certain point of view, there is a cohomology theory on topological spaces.

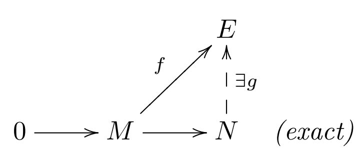
Theorem 3.5.3. *Mod_R has enough injectives. In other words, every R-module can be embedded into an injective R-module.*

Proof. Let $M \in \mathrm{Mod}_R$. Let $M_{\mathbf{Z}}$ be M regarded as a $\mathbf{Z}$-module. Since $\mathrm{Mod}_{\mathbf{Z}}$ has enough injectives, there exists an injective $\mathbf{Z}$-module $\mathscr{E}$ such that we have an injective $\mathbf{Z}$-map $M_{\mathbf{Z}} \hookrightarrow \mathscr{E}$. Let $E = \mathrm{Hom}_{\mathbf{Z}}(R, \mathscr{E})$. We saw in Lecture 9 (see pp.4–5) that E is an R-module. By Problem 4 of Homework 3, we know that E is an injective R-module. We claim there is an embedding $j\colon M \hookrightarrow E$ of M into E. For $m \in M$, set $j(m) = \phi_m$ where $\phi_m\colon R \to \mathscr{E}$ is the $\mathbf{Z}$-map $r \mapsto rm$. Now $j(m) = 0$ if and only if $\phi_m = 0$, and this is so if and only if $rm = 0$ for every $r \in R$. Taking $r = 1$, we get $m = 0$. Thus j is an embedding, and we are done. □

3.6. **Characterising injective modules.** In Subsection 3.1 we gave an alternate definition of an injective module which was obviously equivalent to the one we gave earlier in the course. Here is a theorem giving that and other characterisations.

Theorem 3.6.1. *Let $E \in \mathrm{Mod}_R$. The following are equivalent.*

(a) *E is injective.*
(b) *If M is a submodule of a module N, then every R-map $f\colon M \to E$ can be extended to an R-map $g\colon N \to E$.*
(c) *Given a diagram of R-maps*



with the row exact (as indicated), the broken arrow can be filed to make the diagram commute.

(d) *If E is a submodule of a module M, then E is a direct summand of M, i.e. there exists a submodule T of M such that $M = E \oplus T$.*

Proof. For the equivalence of (a), (b), and (c), see § 3.1. Let us prove (c) implies (d). We know that the broken arrow in the diagram below can be filled, say by the map $\tau\colon M \to E$.

$$\begin{array}{ccccc} & & & & E \\ & & & \overset{\mathbf{1}_E}{\nearrow} & \uparrow \\ 0 & \longrightarrow & E & \longrightarrow & M \end{array} \qquad \text{(exact)}$$

Then, according to Proposition 1.1.3, τ induces a splitting of the exact sequence

$$0 \longrightarrow E \longrightarrow M \longrightarrow M/E \longrightarrow 0.$$

This proves (d).

To prove (d) implies (a), embed E in an injective module $\mathscr{E}$. Then $\mathscr{E} = E \oplus K$ for some R-module K. By Lemma 3.3.1 E must be injective. $\square$