The Construction of Tensor Products

Let A be a ring, and M, N two A-modules. Let F be the free module on the symbols $(m,n) \in M \times N$. In other words

$$F = \bigoplus_{(m,n) \in M \times N} F_{(m,n)}$$

where each $F_{(m,n)} = A$. For good book-keeping we will write $e(m,n)$ for the element $(f_{\mu,\nu})_{(\mu,\nu) \in M \times N} \in F$ where

$$f_{\mu,\nu} = \begin{cases} 0 & \text{if } (\mu,\nu) \neq (m,n) \\ 1 & \text{if } (\mu,\nu) = (m,n). \end{cases}$$

In other words $\{e(m,n)\}_{(m,n)}$ is the standard basis of F.

Let $G \subseteq F$ be the submodule generated by elements of the following form:

(1) $e(m+m', n) - e(m,n) - e(m',n)$, $m, m' \in M$, $n \in N$

(2) $e(m, n+n') - e(m,n) - e(m,n')$, $m \in M$, $n, n' \in N$

(3) $e(am, n) - e(m, an)$, $a \in A$, $m \in M$, $n \in N$

(4) $e(am, n) - a\, e(m,n)$, $a \in A$, $m \in M$, $n \in N$.

For $m \in M$, $n \in N$, let $[m,n]$ be the coset

$$e(m,n) + G \in F/G.$$

Properties (1) — (4) translate to

$$\left.\begin{array}{ll} (a) & [m+m', n] = [m,n] + [m',n] \\ (b) & [m, n+n'] = [m,n] + [m,n'] \\ (c). & [am, n] = [m, an] = a[m,n] \end{array}\right\} \begin{array}{l} a \in A,\ m, m' \in M \\ \text{and } n, n' \in N. \end{array}$$

Note that every element of F/G is of the form

$$\sum_{i=1}^{l} [m_i, n_i], \quad m_i \in M,\ n_i \in N,\ i = 1, \ldots, l.$$

Let

$$B^* : M \times N \longrightarrow F/G$$

be the map $(m,n) \longmapsto [m,n]$. From (a), (b), and (c) above, it is apparent that B^* is A-bilinear. Now suppose for $T \in \mathrm{Mod}_A$ we have an A-bilinear map

$$B : M \times N \longrightarrow T.$$

Define $\varphi_B : F/G \longrightarrow T$ by the formula

$$\varphi_B\left(\sum_i^{1} [m_i, n_i]\right) = \sum_i^{1} B(m_i, n_i).$$

To show this is well-defined, it is enough to show that

$$\sum_i^{1} B(m_i, n_i) = 0$$

for every $\sum_i^{1} [m_i, n_i] \in G$. For this it is enough to prove this for the generators of G. Since B is bilinear this is clear. Thus

$$\begin{tikzcd} M \times N \arrow[d, "B^*"'] \arrow[dr, "B"] & \\ F/G \arrow[r, "\varphi_B"'] & T \end{tikzcd} \qquad (\Delta)$$

commutes. Moreover, if $\psi: F/B \longrightarrow T$ any A-map s.t. $\psi \circ B^* = B$, then $\psi([m,n]) = \psi(B^*(m,n)) = B(m,n)$, and hence ψ and φ_B agree on elements of the form $[m,n]$. By A-linearity, they agree on F/G. Thus φ_B is the only A-map making diagram (Δ) commute.

Conclusion: Tensor products exist with.

$$F/G = M \otimes_A N, \quad B_u = B^*.$$

Remark: The exact "construction" of $M \otimes_A N$ is not important. The role of the above construction is to show that it exists.

Multilinear Maps:

Let $M_1, M_2, \dots, M_d$ be A-modules (A, as usual, a ring). Let $T \in \mathrm{Mod}_A$. A map

$$\circledast : M_1 \times \dots \times M_d \longrightarrow T$$

is said to be A d-linear (or simply d-linear) if the following conditions are satisfied for $m_i, m_i' \in M_i$ and $a \in A$

$$\circledast(m_1, m_2, m_i + a m_i', \dots, m_d)$$

$$= \circledast(m_1, m_2, \dots, m_d) + a\,\circledast(m_1, m_2, \dots, m_{i-1}, m_i', m_{i+1}, \dots, m_d).$$

Note that

$$\circledast(a m_1, m_2, \dots, m_d) = \circledast(0 + a m_1, m_2, \dots, m_d)$$
$$= \circledast(0, m_2, \dots, m_d) + a\,\circledast(m_1, m_2, \dots, m_d)$$

$$= a \circledast (m_1, \dots, m_d).$$

($\circledast(0, m_2, \dots, m_d) = 0$ via the usual tricks.)

Similarly

$$\circledast(m_1, m_2, \dots, m_{i-1}, a m_i, m_{i+1}, \dots, m_d)$$
$$= a \circledast (m_1, \dots, m_d)$$

Example: Let $M_i \in A^d$, $i = 1, \dots, d$.

Write elements of A^d as column vectors. Define

$$\Delta : \underbrace{A^d \times \dots \times A^d}_{d\text{-times}} \longrightarrow A$$

by the rule

$$\Delta(C_1, \dots, C_d) = \det [C_1 \, C_2 \dots C_d]$$

where, as usual, $[C_1 \, C_2 \dots C_d]$ is the $d \times d$ matrix whose i^{th} column is C_i.

The tensor product of the M_i's is an A d-linear map

$$\otimes_u : M_1 \times \dots \times M_d \longrightarrow M_1 \otimes_A \dots \otimes_A M_d$$

which has the obvious universal property for A d-linear maps, namely if $T \in \mathrm{Mod}_A$ and $\circledast : M_1 \times \dots \times M_d \longrightarrow T$ is d-linear, then $\exists!$ A-module map $\varphi_{\circledast} : M_1 \otimes_A \dots \otimes_A M_d \longrightarrow T$ s.t. TFDC.

$$\begin{tikzcd}
M_1 \times \dots \times M_d \arrow[d, "\otimes_u"'] \arrow[dr, "\circledast"] & \\
M_1 \otimes_A \dots \otimes_A M_d \arrow[r, "\varphi_{\circledast}"'] & T
\end{tikzcd}$$

Suppose $d = 3$. It is quite easy to see that $(M_1 \otimes_A M_2) \otimes_A M_3$ as well as $M_1 \otimes_A (M_2 \otimes_A M_3)$ satisfy the required universal property for 3-linear maps, by using the universal property of bilinear maps repeatedly.

Suppose $\sigma \in S_d$ is a permutation on $\{1, \dots, d\}$. It is clear that every multilinear map $M_1 \times \dots \times M_d \longrightarrow T$ gives rise to a unique multilinear map $M_{\sigma(1)} \times \dots \times M_{\sigma(d)} \longrightarrow T$, and from here it is not hard to see that

$$M_1 \otimes_A \dots \otimes M_d \xrightarrow{\sim} M_{\sigma(1)} \otimes_A \dots \otimes_A M_{\sigma(d)}.$$

Summary:

1. $(M_1 \otimes_A M_2) \otimes_A M_3 \xrightarrow{\sim} M_1 \otimes_A (M_2 \otimes M_3) \xrightarrow{\sim} M_1 \otimes_A M_2 \otimes_A M_3.$

2. $M_1 \otimes_A \cdots \otimes_A M_d \xrightarrow{\sim} M_{\sigma(1)} \otimes_A \cdots \otimes_A M_{\sigma(d)} \quad \forall\, \sigma \in S_d.$

Among other things, you can put brackets pretty much where you wish in $M_1 \otimes_A M_2 \otimes_A \cdots \otimes_A M_d$.

$$((M_1 \otimes_A M_2) \otimes_A M_3) \otimes_A M_4 = M_1 \otimes_A (M_2 \otimes_A M_3) \otimes_A M_4$$
$$= M_1 \otimes_A ((M_2 \otimes_A M_3) \otimes_A M_4) \quad \text{etc, etc.}$$

You can permute the M_i's too (for fun).

$\mathrm{Hom}_A(M, N)$: Let $M, N \in \mathrm{Mod}_A$. Let $\mathrm{Hom}_A(M,N)$ be the set of A-linear maps from M to N. In other words

$$\mathrm{Hom}_A(M,N) = \{ \varphi : M \longrightarrow N \mid \varphi \text{ is } A\text{-linear} \}.$$

Pointwise addition gives an abelian group structure on $\mathrm{Hom}_A(M,N)$. A scalar multiplication

$$A \times \mathrm{Hom}_A(M,N) \longrightarrow \mathrm{Hom}_A(M,N)$$

can be defined on $\mathrm{Hom}_A(M,N)$ by the formula

$$(a, \varphi) \longmapsto \varphi \circ \mu_a \quad , \; a \in A, \; \varphi \in \mathrm{Hom}_A(M,N)$$

where, as always, μ_a is the "multiplication by a" endomorphism on M, i.e. $\mu_a : M \longrightarrow M$ is the map $m \mapsto am$, $m \in M$. If we also denote the "multiplication by a" map on N by μ_a, then note that $\varphi \circ \mu_a = \mu_a \circ \varphi$. In any case, this gives us a scalar multiplication on $\mathrm{Hom}_A(M,N)$, a fact that is readily checked.

Thus, $\mathrm{Hom}_A(M,N)$ is an A-module.

Next, suppose $f : A \longrightarrow B$ is a ring homomorphism. Then every B-module M can be regarded as an A-module, where the scaler multiplication is

$$a \cdot m := f(a) m \qquad a \in A, \; m \in M.$$

If, as above, $M \in \mathrm{Mod}_B$ and $N \in \mathrm{Mod}_A$, then regarding M as an A-module, we have an A-module $\mathrm{Hom}_A(M,N)$. We claim that $\mathrm{Hom}_A(M,N)$ is actually a B-module. The scalar multiplication

$$B \times \mathrm{Hom}_A(M,N) \longrightarrow \mathrm{Hom}_A(M,N)$$

is the map

$$(b, \phi) \longmapsto \phi \circ \mu_b \quad , \qquad b \in B, \ \phi \in \mathrm{Hom}_A(M, N),$$

where $\mu_b : M \longrightarrow M$ has its usual meaning.

Hom-$\otimes$ adjointness: Suppose $M, N, T \in \mathrm{Mod}_A$. Next lecture we will show a natural isomorphism

$$\mathrm{Hom}_A(M \otimes_A N, T) \xrightarrow{\sim} \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, T))$$

The map is

$$B \longmapsto m \longmapsto (n \longmapsto B(m, n))$$

where $B : M \times N \longrightarrow T$ is a bilinear map (and hence regarded, by the universal property $\otimes$-products, as an element of $\mathrm{Hom}_A(M \otimes_A N, T)$).