Feb 3, 2022 — Lecture 8 — Alg II

Throughout A is a ring. Unless otherwise stated $X = \operatorname{Spec} A$.

Proposition: (a) Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be prime ideals in A and $\mathfrak{a}$ an ideal contained in $\bigcup_{i=1}^n \mathfrak{p}_i$. Then $\mathfrak{a} \subseteq \mathfrak{p}_i$ for some i.
(b) Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be ideals in A, $\mathfrak{p} \in \operatorname{Spec}(A)$ be such that $\mathfrak{p} \supseteq \bigcap_{i=1}^n \mathfrak{a}_i$. Then $\mathfrak{p} \supseteq \mathfrak{a}_i$ for some i. If $\mathfrak{p} = \bigcap_{i=1}^n \mathfrak{a}_i$ then $\mathfrak{p} = \mathfrak{a}_i$ for some i.

Proof: (a) We'll prove the contrapositive. In other words we'll show that:

$$\mathfrak{a} \not\subseteq \mathfrak{p}_i \text{ for any } i \in \{1, \ldots, n\} \Rightarrow \mathfrak{a} \not\subseteq \bigcup_{i=1}^n \mathfrak{p}_i.$$

The above statement is clearly true for $n = 1$. Assume it is true for $n = k-1$ for some $k > 1$. We'll show it is true for $n = k$.

By induction hypothesis for each $i = 1, \ldots, k$ we have

$$\mathfrak{a} \not\subseteq \mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_{i-1} \cup \mathfrak{p}_{i+1} \cup \cdots \cup \mathfrak{p}_k.$$

This means there exists, for each $i \in \{1, \ldots, k\}$, an element $a_i \in \mathfrak{a}$ s.t. $a_i \notin \mathfrak{p}_j$ for $j \neq i$. If $a_i \notin \mathfrak{p}_i$ for any i we are done, for then $a_i \notin \bigcup_{j=1}^k \mathfrak{p}_j$. If $a_i \in \mathfrak{p}_i$ for each i in $\{1, \ldots, k\}$, then consider the element

$$x = \sum_{j=1}^k a_1 a_2 \ldots a_{j-1} a_{j+1} \cdots a_k.$$

Clearly, $x \in \mathfrak{a}$. Equally clearly $x \notin \mathfrak{p}_j$ for any j. So $x \notin \bigcup_{j=1}^k \mathfrak{p}_j$. Hence we are done.

(b) Suppose $\mathfrak{p} \not\supseteq \mathfrak{a}_i$ for any i. Then for each i $\exists\, x_i \in \mathfrak{a}_i$ s.t. $x_i \notin \mathfrak{p}$. Then $x_1 \ldots x_n \in \bigcap_{i=1}^n \mathfrak{a}_i$ and $x_1 \ldots x_n \notin \mathfrak{p}$, i.e. $\mathfrak{p} \not\supseteq \bigcap_{i=1}^n \mathfrak{a}_i$. This proves the first part of (b). Now suppose $\mathfrak{p} = \bigcap_{i=1}^n \mathfrak{a}_i$. Then $\mathfrak{p} \subseteq \mathfrak{a}_j$ for every j. On the other hand, we know $\mathfrak{p} \supseteq \mathfrak{a}_i$ for some i. It follows that

$\mathfrak{p} = \mathfrak{a}_i$ for this i. //

Topological properties of Spec (A):

In this section
$$X := \text{Spec}(A).$$

Recall, if $\mathfrak{a} \subseteq A$ is an ideal then
$$V(\mathfrak{a}) := \{\mathfrak{p} \in X \mid \mathfrak{p} \supseteq \mathfrak{a}\}.$$

1. $$V\left(\bigcap_{i=1}^{n} \mathfrak{a}_i\right) = \bigcup_{i=1}^{n} V(\mathfrak{a}_i).$$

Pf: If $\mathfrak{p} \in V(\bigcap_{i=1}^{n} \mathfrak{a}_i)$ then $\mathfrak{p} \supseteq \bigcap_{i=1}^{n} \mathfrak{a}_i$, whence $\mathfrak{p} \supseteq \mathfrak{a}_i$ for some $i \in \{1, \ldots, n\}$, i.e. $\mathfrak{p} \in V(\mathfrak{a}_i)$ for some $1 \leq i \leq n$. So L.S. $\subseteq$ R.S.

Conversely, suppose $\mathfrak{p} \in \bigcup_{i=1}^{n} V(\mathfrak{a}_i)$. Then $\mathfrak{p} \in V(\mathfrak{a}_i)$ for some i, i.e. $\mathfrak{p} \supseteq \mathfrak{a}_i$ for some i, i.e. $\mathfrak{p} \supseteq \bigcap_{i=1}^{n} \mathfrak{a}_i$. Thus $\mathfrak{p} \in$ L.S. Thus R.S. $\subseteq$ L.S. //

2. Suppose $\{\mathfrak{a}_\alpha \mid \alpha \in \Sigma\}$ is a family of ideals. Then
$$V\left(\sum_\alpha \mathfrak{a}_\alpha\right) = \bigcap_{\alpha \in \Sigma} V(\mathfrak{a}_\alpha)$$

Pf:

Suppose $\mathfrak{p} \in \bigcap_\alpha V(\mathfrak{a}_\alpha)$. Then $\mathfrak{p} \supseteq \mathfrak{a}_\alpha$ for every $\alpha \in \Sigma$, and hence $\mathfrak{p}$ contains the smallest ideal containing all the $\mathfrak{a}_\alpha$, i.e. $\mathfrak{p} \supseteq \sum_\alpha \mathfrak{a}_\alpha$, i.e. $\mathfrak{p} \in V(\sum_\alpha \mathfrak{a}_\alpha)$

Conversely, suppose $\mathfrak{p} \in V(\sum_\alpha \mathfrak{a}_\alpha)$. Then
$$\mathfrak{p} \supseteq \sum_\alpha \mathfrak{a}_\alpha.$$

This means $\mathfrak{p} \supseteq \mathfrak{a}_\alpha$ for every α. Hence $\mathfrak{p} \in V(\mathfrak{a}_\alpha)$ for every α. //

The above properties show that $X = \operatorname{Spec} A$ is a topological with its closed sets being $V(\mathfrak{a})$, $\mathfrak{a}$ an ideal A. The only things to be verified are that ϕ and X are in this collection $\{V(\mathfrak{a}) \mid \mathfrak{a} \text{ ideal of } A\}$. Now,

$$\phi = V(\langle 1 \rangle), \quad X = V(\langle 0 \rangle).$$

So ...

3. Recall, if U is an open set in X, then $U = X \setminus V(\mathfrak{a})$, where $\mathfrak{a}$ is an ideal. Now

$$\mathfrak{a} = \sum_{f \in \mathfrak{a}} \langle f \rangle$$

Hence $$V(\mathfrak{a}) = \bigcap_{f \in \mathfrak{a}} V\langle f \rangle$$

i.e.

$$U = \bigcup_{f \in \mathfrak{a}} (X \setminus V(\langle f \rangle)).$$

For $f \in A$, denote $X \setminus V(\langle f \rangle)$ by $D(f)$

$$D(f) = X \setminus V(\langle f \rangle).$$

Therefore $$U = \bigcup_{f \in \mathfrak{a}} D(f).$$

Note also that

$$D(f) \cap D(g) = D(fg). \qquad \text{(check!!)}$$

This means
$$\mathcal{B} = \{D(f) \mid f \in A\}$$

is a base for the topology on X. This is called the standard basis for the topology on X.

4. Let $\phi : A \to B$ be a ring map. Recall that $\phi^{-1}(P)$ is a prime ideal in A whenever P is a prime ideal of B. This gives a map

$$
\begin{aligned}
{}^{a}\phi : \operatorname{Spec} B &\longrightarrow \operatorname{Spec} A \\
P &\longmapsto \phi^{-1}(P).
\end{aligned}
$$

It is easy to see that for $f \in A$,

$$({}^{a}\phi)^{-1}(D(f)) = D(\phi(f))$$

This proves that ${}^{a}\phi : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ is a continuous map.

Check the following:

If $A \xrightarrow{\phi} B \xrightarrow{\psi} C$ is a pair of ring maps, then the following diagram commutes:

$$
\begin{tikzcd}
 & & \operatorname{Spec} A \\
\operatorname{Spec} C \arrow[urr, "{}^{a}(\psi\circ\phi)"] \arrow[drr, "{}^{a}\psi"'] & \circlearrowright & \\
 & & \operatorname{Spec} B \arrow[uu, "{}^{a}\phi"']
\end{tikzcd}
$$

In other words: "Spec is a functor from the category of rings to the category of topological spaces".

This is NOT a faithful functor.

Take $A = k[T]/\langle T^2 \rangle$, T a variable, k a field.

Let $\varepsilon = T + \langle T^2 \rangle$. Then $A = k[\varepsilon]$, $\varepsilon^2 = 0$.
$A = \{a + b\varepsilon \mid a, b \in k\}$. Since $b\varepsilon$ is nilpotent, if $a \neq 0$, then $a + b\varepsilon$ is a unit. This means either an element is nilpotent or is a unit.
Let $\mathfrak{m} = \langle \varepsilon \rangle$. Since $k \simeq A/\mathfrak{m}$, $\mathfrak{m}$ is a max'l ideal. And from our argument, it is the

only proper ideal of A. This shows that

$$\mathrm{Spec}(A) = \{\mathfrak{m}\}.$$

Now consider the map

$$\phi : A \twoheadrightarrow A/\mathfrak{m} = k.$$

$$^a\phi : \mathrm{Spec}(k) \longrightarrow \mathrm{Spec}(A)$$

is a homeomorphism since both topological spaces are singleton sets. However ϕ is not an isomorphism of rings.

5. Let $f \in A$, and in observation 4, let $B = A_f$, i.e. $B = S^{-1}A$, where $S = \{f^n \mid n \geq 0\}$. Recall we have a localization map

$$A \longrightarrow A_f.$$

The resulting map

$$\mathrm{Spec}(A_f) \longrightarrow \mathrm{Spec}(A)$$

is injective, its image is $D(f)$, and finally the continuous map

$$\mathrm{Spec}(A_f) \longrightarrow D(f)$$

is a homeomorphism.