Lemma 2 (Zariski's version of the Nullstellensatz): (Zariski's Lemma) Let $k \to L$ be an extension of fields such that L is of finite type over k. Then L is a finite dim'l vector space over k.

Proof:

First note that $F = k[\theta_1, \ldots, \theta_n]$ is a field extension of k with each θ_i algebraic over k, then $k[\theta_1, \ldots, \theta_n]$ is finite dim'l as a k-vector space.

Case $n=1$: $F = k[\theta]$. Let $p \in k[X]$ — (the poly. ring in one var $/k$) be the irreducible poly of θ over k.

Then $$F \simeq \frac{k[X]}{\langle p \rangle}.$$ (e.g. $\mathbb{C} = \frac{\mathbb{R}[x]}{\langle x^2+1 \rangle}$)

with θ mapping to $X + \langle p \rangle$. Let $d = \deg p$.

Then $1, \theta, \theta^2, \ldots, \theta^{d-1}$ forms a basis of F over k (check).

If $n > 1$, then apply the above to F over $k[\theta_1, \ldots, \theta_{n-1}]$.

In view of this, it is enough for us to prove that every element in L is algebraic over k.

Let $\xi_1, \ldots, \xi_m$ be k-algebra generators of L. Suppose L has transcendental elements. Then one ξ_i's is transcendental. We can re-order the ξ_i's so that ξ_1 is transcendental over k

ξ_2 is transcendental over $k(\xi_1)$

$\vdots$

ξ_d is transcendental over $k(\xi_1, \ldots, \xi_{d-1})$

and $\xi_{d+1}, \ldots, \xi_m$ are algebraic over

$$K = k(\xi_1, \ldots, \xi_d)$$

$$\xi_1, \ldots, \xi_d, \Big| \, \xi_{d+1} \ldots, \xi_m$$

The situation is as follows:

$$\begin{array}{l} L = K[\xi_{d+1}, \ldots, \xi_m] \\ \cup \quad \leftarrow \text{algebraic} \\ K = k(\xi_1, \ldots, \xi_d) \\ \cup \quad \leftarrow \text{transcendental.} \\ k \end{array}$$

From our earlier observations, L is a finite dim'l K-vector space. By Artin-Tate K is of finite type over k (since L is of finite type over k, and k is Noetherian). In particular K is of finite type over $k(\xi_1, \ldots, \xi_{d-1})$. In other words

$$K = k(\xi_1, \ldots, \xi_d) = k(\xi_1, \ldots, \xi_{d-1})(\xi_d)$$

is of finite type over $k(\xi_1, \ldots, \xi_{d-1})$. This is not possible from what we proved in the last lecture since ξ_d is transcendental over $k(\xi_1, \ldots, \xi_{d-1})$. //

Corollary: Suppose k is an algebraically closed field and L a field ext$^{\underline{n}}$ of k which is of finite type over k. Then the ext$^{\underline{n}}$ $k \longrightarrow L$ is an isomorphism.

Proof:

We've shown that L is algebraic over k, and we know that k is alg. closed. //

Hilbert's Nullstellensatz:

The thm says the "zero set" of a proper ideal of a polynomial ring over an algebraically closed field is non-empty.

Suppose k is algebraically closed.

Let $\vec{x} = (x_1, \ldots, x_n) \in k^n$. Let $A = k[X_1, \ldots, X_n]$ be the poly. ring $/k$ in n-variables.

Let $m_{\vec{x}} = \langle X_1 - x_1, X_2 - x_2, \ldots, X_n - x_n \rangle$

Let

$$f \in A = k[X_1, \ldots, X_n].$$

We can write f as a polynomial in $X_1 - x_1, \ldots, X_n - x_n$:

$$f = f(X_1, \ldots, X_n) = f(X_1 - x_1 + x_1, \ldots, X_n - x_n + x_n)$$

$$= c_0 + \sum c_{\mu_1 \ldots \mu_n} (X_1 - x_1)^{\mu_1} \cdots (X_n - x_n)^{\mu_n}$$

Clearly $f(x_1, \ldots, x_n) = c_0$.

In other words

$$f = f(x_1,\dots,x_n) + (X_1-x_1)g_1 + \dots + (X_n-x_n)g_n,$$
$$g_1,\dots,g_n \in k[X_1,\dots,X_n]$$

Note $(X_1-x_1)g_1 + \dots + (X_n-x_n)g_n \in \mathfrak{M}_x$.

Consider the surjective ring map

$$A = k[X_1,\dots,X_n] \twoheadrightarrow k$$

given by $\quad f \longmapsto f(x_1,\dots,x_n)$ $\Big\}$ The "substitution map"

From what we've seen above, the kernel is $\mathfrak{M}_x$.

$$\text{So} \quad A/\mathfrak{M}_x \xrightarrow{\sim} k$$

where $\mathfrak{M}_x$ is a maximal ideal.

Suppose $\mathfrak{M}$ is a maximal ideal of A. Then we have a pair of ring maps

$$k \underset{\text{usual inclusion}}{\longrightarrow} A = k[X_1,\dots,X_n] \underset{\substack{\text{natural surjection} \\ f \mapsto f+\mathfrak{M}}}{\twoheadrightarrow} A/\mathfrak{M} =: L$$

Clearly L is of finite type over k, being generated by the images of $X_1,\dots,X_n$.

By Zariski's Lemma, since k is alg. closed, $k \longrightarrow L$ is an isomorphism. Let $\psi : k \longrightarrow L$ be the above isomorphism, and $\phi : L \longrightarrow k$ its inverse. Let $\xi_i := \psi(X_i) \in L$, $i = 1,\dots,n$, and let $x_i := \phi(\xi_i)$, $i = 1,\dots,n$.

$$\begin{array}{rcl} & & k \\ & & \wr\!\! \mid \\ k[X_1,\dots,X_n] & \twoheadrightarrow & L \\ X_i & \longmapsto & \xi_i \\ x_i & \hookleftarrow & \\ X_i - x_i & \longmapsto & \xi_i - \xi_i \\ & & = 0 \end{array}$$

Now $X_i - x_i$ maps to $\xi_i - \xi_i = 0$ for $i = 1, \dots, n$ under the natural map $A = k[X_1, \dots, X_n] \twoheadrightarrow L = A/\mathfrak{m}$. Hence $X_i - x_i \in \mathfrak{m}$ $\forall\, i$.

In particular, with $\vec{x} = (x_1, \dots, x_n)$
$$\mathfrak{m}_{\vec{x}} \subseteq \mathfrak{m}.$$
Now $\mathfrak{m}_{\vec{x}}$ is a max'l ideal (we just proved this moments ago!). So
$$\mathfrak{m}_{\vec{x}} = \mathfrak{m}.$$

Theorem (Hilbert's Nullstellensatz) Let k be an algebraically closed field, $A = k[X_1, \dots, X_n]$ a polynomial ring in n-variables over k with $n \geq 1$, and I a proper ideal of A. Then:

(a) There exists at least one point $\vec{x} = (x_1, \dots, x_n) \in k^n$ such that $f(\vec{x}) = 0$ for all $f \in I$.

(b) If $\mathfrak{m}$ is a maximal ideal of A, then there exists a unique point $\vec{x} = (x_1, \dots, x_n) \in k^n$ such that $\mathfrak{m} = \langle X_1 - x_1, \dots, X_n - x_n \rangle$.

Proof:

We have already proved (b). To prove (a), pick a max'l ideal $\mathfrak{m}$ containing I. From (b), $\mathfrak{m} = \langle X_1 - x_1, \dots, X_n - x_n \rangle$ for a unique $(x_1, \dots, x_n) \in k^n$. It follows that $I \subseteq \langle X_1 - x_1, \dots, X_n - x_n \rangle$, whence $f(x_1, \ldots, x_n) = 0$ for all $f \in I$. //

Remarks: 1. Recall if R is the ring of continuous functions on $I = [a,b] \subseteq \mathbb{R}$, then the maximal ideals of R are in bijective correspondence with points of I: If $x \in I$, set M_x equal to the set of cts functions vanishing at x. Then M_x is max'l and $x \longleftrightarrow M_x$ is the bijective correspondence

2. The statement of the above theorem is often called the "weak nullstellensatz" though it is equivalent to all other forms of the nullstellensatz.

Theorem (Hilbert's Nullstellensatz - II): Let k be an alg. closed field and I a proper ideal of $A = k[X_1, \ldots, X_n]$ where $n \geq 1$. Let

$$V(I) = \{\vec{x} \in k^n \mid g(\vec{x}) = 0 \ \forall g \in I\}$$

If $f \in A$ is such that $f(\vec{x}) = 0 \ \forall \vec{x} \in V(I)$, then there exists a positive integer m such that $f^m \in I$. In other words $f \in A$ vanishes at all points of $V(I)$ if and only $f \in \sqrt{I}$.

Proof:

Suppose $f \in A$ such that $f^m \notin I$ for any positive integer m. By what we've proved in earlier lectures, there exists a prime ideal $\mathfrak{p}$ of A containing I such that $f \notin \mathfrak{p}$. This is because

$$\sqrt{I} = \bigcap_{\substack{\mathfrak{p} \supseteq I \\ \mathfrak{p} \in \mathrm{Spec}(A)}} \mathfrak{p}$$

(Recall: $\sqrt{I} := \{ g \in A \mid g^m \in I \text{ for some } m \geq 1 \}$)

Let

$$B = (A/\mathfrak{p})_f .$$

Then B is of finite type over k.

<u>Reason:</u> $B = \left((A/\mathfrak{p})[Y] \right) / \langle fY - 1 \rangle$.

Let $\mathfrak{M}$ be a max'l ideal of B and for $i = 1, \dots, n$, let x_i be the image of X_i under the composite $A \twoheadrightarrow A/\mathfrak{p} \longrightarrow B \longrightarrow B/\mathfrak{M}$. By Zariski's lemma, x_i may be regarded as an element of k. It is clear (?) that $f(x_1, \dots, x_n) \neq 0$. //