Recall from last time: A ring, then $\mathrm{Spec}(A)$ is the set of prime ideals of A. (If $A=0$, then $\mathrm{Spec}(A)=\phi$.)
If $\mathfrak{a}$ is an ideal of A, then

$$V(\mathfrak{a}) := \{\mathfrak{p}\in \mathrm{Spec}(A) \mid \mathfrak{p}\supseteq \mathfrak{a}\}$$

Turns out (will be proven later):

$$V\Big(\sum_\alpha \mathfrak{a}_\alpha\Big) = \bigcap_\alpha V(\mathfrak{a}_\alpha)$$

and

$$V\Big(\bigcap_{i=1}^n \mathfrak{a}_i\Big) = \bigcup_{i=1}^n V(\mathfrak{a}_i)$$

$$V(\langle 0\rangle) = \mathrm{Spec}(A)$$

$$V(A) = \phi$$

Can define a topology on $\mathrm{Spec}(A)$ by declaring the $V(\mathfrak{a})$'s to be the closed sets in our topology.

Also for $f\in A$, if $D(f) = \{\mathfrak{p}\in \mathrm{Spec}\,A \mid f\notin\mathfrak{p}\}$ then $D(f)$ is an open set in $\mathrm{Spec}(A)$, $\mathcal{B} = \{D(f) \mid f\in A\}$ is a basis for the topology on $\mathrm{Spec}(A)$ and finally $D(f)$ can be identified (as a top. space) with $\mathrm{Spec}(A_f)$.

Hilbert's Nullstellensatz

Artin-Tate:

Theorem (Artin-Tate Lemma): Let $A\to B\to C$ be ring homomorphisms with $B\to C$ an inclusion. Suppose A is Noetherian and C is a finite type algebra over A. If C

is finitely generated as a module over B, then B is also of finite type over A.

Proof:

Let $c_1, \dots, c_n$ be a set of generators for C as an A-algebra. Let $r_1, \dots, r_m$ be generators of C as a B-module.

~~Pick $r_1, \dots, r_m$ so that $\{r_1, \dots, r_m\} \supset \{c_1, \dots, c_n\}$.~~

We have $b_{ij} \in B$, $1 \le i \le n$, $1 \le j \le m$

$$c_i = \sum_{j=1}^{m} b_{ij} r_j \qquad i = 1, \dots, n. \qquad \text{———} (*)$$

We also have $\beta_{ijk} \in B$, $i,j,k = 1, \dots, m$ such that

$$r_i r_j = \sum_{k=1}^{m} \beta_{ijk} r_k \qquad i,j = 1, \dots, m. \qquad \text{———} (**)$$

Let B_0 be the A-algebra generated by the b_{ij} and the β_{ijk}. Then B_0 is of finite type over A.

$$\begin{array}{l} C \\ \cup \\ B \\ \cup \\ B_0 \leftarrow \text{Noeth. by HBT.} \\ \uparrow \\ A \leftarrow \text{Noetherian ring} \end{array}$$

Since A is Noetherian, by Hilbert's Basis Theorem, B_0 is also Noetherian.

We claim that C is finitely generated as a B_0-module.

Indeed, given an element $c \in C$, it can be written a polynomial expression $p(c_1, \dots, c_n)$ in $c_1, \dots, c_n$ with coefficients in A, and whence as a polynomial in $r_1, \dots, r_m$ with coefficients in the A-algebra generated by the b_{ij} defined above. (Use $(*)$)

Now any monomial in the r_j's can be re-written as a linear combination in the r_j with coefficients coming B_0 (use (**)). This proves the claim for we have just shown that $r_1, \dots, r_m$ are B_0-module generators of C.

Now, as observed earlier, B_0 is Noetherian (by HBT), and since C is f.g. as a B_0-module, it is a Noetherian B_0-module (by the HBT for modules).
It follows that B is a f.g. B_0-module.
Let $x_1, \dots, x_d$ be B_0-module generators of B.
So any element $b \in B$ can be written as

$$b = \sum_{i=1}^{d} \Delta_i x_i \quad , \qquad \Delta_i \in B_0.$$

Now Δ_i is a polynomial expression over A in the b_{ij}, β_{ijk}, and hence b can be written as a polynomial expression over A in $\{b_{ij}, \beta_{ijk}, x_\ell, \dots\}$.
In other words B is of finite type over A. //

Two lemmas:

Lemma 1: Suppose k is a field, and t is element (in a field extension of k) which is transcendental over k.
Then $k(t)$ is not of finite type over $k[t]$.

Explanation of notations: Let $A \to B$ be a ring map, and $b_1, \dots, b_n$ elements in B. By $A[b_1, \dots, b_n]$ we mean the subring of B generated over A by $b_1, \dots, b_n$, i.e.,

the image of the ring map
$$A[X_1,\dots,X_n] \longrightarrow B$$
given by $p \longmapsto p(b_1,\dots,b_n)$
where $A[X_1,\dots,X_n]$ is the poly. ring in n-variables over A.

By $k(t)$ we mean the field of fractions of $k[t]$, i.e. "rational functions" of the form $p(t)/q(t)$, $q(t) \neq 0$, p, q polynomials in t with coeffs in k.

$$\begin{tikzcd} k(t) \arrow[dr, bend left, hook] & \\ k[t] \arrow[u, hook] \arrow[r, hook] & L \\ k \arrow[u, hook] & \end{tikzcd}$$

Proof of Lemma: Since t is transcendental over k, $k[t]$ is the polynomial ring in one variable over k. Suppose $k(t)$ is of finite type over the polynomial ring $A = k[t]$. Let $b_1,\dots,b_n \in k(t)$ be A-algebra generators of $k(t)$, say $b_i = \dfrac{a_i(t)}{q_i(t)}$ where $a_i(t)$ and $q_i(t)$ are in $A = k[t]$, with $q_i(t)$ non-zero. Since $k[t]$ has an infinite number of irreducible polynomials, we can find an irreducible polynomial $p(t) \in k[t]$ such that $p(t)$ is not a factor of any of the $q_i(t)$, $i = 1,\dots,n$. Since $b_1,\dots,b_n$ generate $k(t)$ as an A-algebra, therefore $\frac{1}{p}$ is a polynomial expression in $b_i = \dfrac{a_i(t)}{q_i(t)}$, $i = 1,\dots,n$ with coefs in A. This is not possible since $p(t)$ does not divide any of the $q_i(t)$. Thus $k(t)$ is not of finite type over A.

$$\frac{1}{p} = \sum \alpha_{j_1 \cdots j_n}(t) \left(\frac{a_1(t)}{q_1(t)}\right)^{j_1} \cdots \left(\frac{a_n(t)}{q_n(t)}\right)^{j_n}$$

(Zariski's Lemma)

Lemma 2 (Zariski's version of the Nullstellensatz): Let $k \to L$ be an extension of fields such that L is of finite type over k. Then L is a finite dim'l vector space over k.