Jan 25, 2022 Lecture 5 Alg II

Notation: A ring. $\mathrm{Spec}(A) := \left\{ \mathfrak{p} \;\middle|\; \begin{array}{l}\mathfrak{p} \text{ a prime ideal} \\ \text{in } A\end{array} \right\}$

The "prime spectrum" of A
or simply the "spectrum" of A.

Last time: The nilradical of $A = \bigcap_{\mathfrak{p} \in \mathrm{Spec} A} \mathfrak{p}$.

$\sqrt{0}$.

Definition: Let A be a ring, $I \subseteq A$ an ideal. The radical of I, denoted $\sqrt{I}$, is

$$\sqrt{I} = \{ a \in A \mid a^n \in I \text{ for some } n \geq 0 \}.$$

Note: $I \subseteq \sqrt{I}$.

Remarks:

1. $\sqrt{I}$ is the unique ideal corresponding to the nilradical of A/I.

$$\sqrt{I} \subseteq A \longleftrightarrow \sqrt{0} \subseteq A/I.$$

2. It follows

$$\sqrt{I} = \bigcap_{\substack{\mathfrak{p} \in \mathrm{Spec} A \\ \mathfrak{p} \supseteq I}} \mathfrak{p}.$$

3. If $\mathfrak{p} \in \operatorname{Spec} A$, then there is a maximal $\mathfrak{M}$ of A containing $\mathfrak{p}$. To see this, note that we proved $A/\mathfrak{p}$ has a maximal ideal (Zorn's Lemma).

4. If I is an ideal of A, then $\exists\ \mathfrak{p} \in \operatorname{Spec} A$ s.t. $\mathfrak{p} \supseteq I$. Once again, pick a prime ideal in A/I and argue as in 3.

Definition: Let A be a ring. The Jacobson radical of A, denoted either $\operatorname{rad}(A)$ or $J(A)$, is the intersection of all maximal ideals of A.

Note: $J(A) \supseteq$ nilradical of $A = \sqrt{0}$.

Remark: Let A be a ring. If $a \in A$ is a unit, then $\exists\ b \in A$ s.t. $ab = 1$, whence $\langle a \rangle = A$. In particular a does not lie any proper ideal of A; and hence in no maximal ideal of A. On the other hand if a is a non-unit, then $\langle a \rangle \subsetneq A$, and hence a lies in some maximal ideal of A.

Conclusion: Let $S = \bigcup_{\mathfrak{M} \in \operatorname{Max}(A)} \mathfrak{M}$, then

$\operatorname{Max}(A)$ ← the set of maximal ideals of A.

$$A \setminus S = \text{set of units of } A.$$

Lemma: Let A be a ring and a an element of $J(A)$. Then $1+a$ is a unit in A. More generally, if u is a unit in A, then $u+a$ is a unit in A.

Proof:

Suppose $u+a$ is not a unit. Then, by the Remark above, $u+a \in \mathfrak{M}$ for some max'l ideal $\mathfrak{M}$ of A. Since $a \in J(A)$, a must lie in $\mathfrak{M}$, whence $u \in \mathfrak{M}$, a contradiction, since $\mathfrak{M} \subsetneq A$. //

Remark: In particular, if a is nilpotent, then $u+a$ is a unit for every unit u in A. A simpler proof of this is:

Recall the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots$$

If x is nilpotent, this actually makes sense, since $0 = x^{n+1} = x^{n+2} = \dots$ for some $n \geq 1$, and one checks easily that

$$(1-x)(1+x+x^2+\dots+x^n) = 1.$$

From here to show $u+x$ is a unit is easy:

$$u+x = u(1+u^{-1}x) = u[1-\underbrace{(-u^{-1}x)}_{\text{nilpotent.}}]$$

Exercise: Suppose $p = a_0 + a_1 x + \dots + a_n x^n \in A[x]$. Show that p is a unit if and only if $a_1, a_2, \dots, a_n$ are nilpotent in A and a_0 is a unit in A.

The determinant Trick and Nakayama's Lemma:

Let A be a ring, $M \in \mathrm{Mod}_A$, M f.g. as an A-module and I an ideal of A.

Set

$$M^d = \underbrace{M \oplus \dots \oplus M}_{d\text{-times}}$$

Write elements of M^d as columns: $\begin{bmatrix} m_1 \\ \vdots \\ m_d \end{bmatrix}$.

can be identified

Note that $\mathrm{End}_A(M^d) = M_{d\times d}(\mathrm{End}(M))$.

$d \times d$ matrices.

If

$$\Phi : M \oplus \dots \oplus M \longrightarrow M \oplus \dots \oplus M$$

is an A-map, then Φ can be represented as

$$\Phi = \begin{pmatrix} \phi_{11} & \dots & \phi_{1d} \\ \phi_{21} & \dots & \phi_{2d} \\ \vdots & & \\ \phi_{d1} & \dots & \phi_{dd} \end{pmatrix}, \quad \phi_{ij} \in \mathrm{End}(M)$$

with

$$\Phi \begin{pmatrix} m_1 \\ \vdots \\ m_d \end{pmatrix} = \begin{pmatrix} \sum_j \phi_{1j} m_j \\ \vdots \\ \sum_j \phi_{dj} m_j \end{pmatrix}$$

Now suppose M is finitely generated say $m = \langle m_1, \dots, m_e \rangle$.

Suppose further
$$M = IM.$$

Then we have elements $a_{ij} \in I$ $1 \le i, j \le e$ such that
$$m_i = \sum_{j=1}^{e} a_{ij} m_j \qquad \text{———— } (*)$$

Regard each a_{ij} as an element of $\mathrm{End}(M)$ via "multiplication by a_{ij}". We have a map

$$M^e \xrightarrow{\Phi} M^e$$
$$\begin{bmatrix} x_1 \\ \vdots \\ x_e \end{bmatrix} \longmapsto \begin{pmatrix} a_{ij} \end{pmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_e \end{bmatrix}$$

Note $\quad (I - \Phi) \begin{bmatrix} m_1 \\ \vdots \\ m_e \end{bmatrix} = 0. \qquad$ (from $(*)$)

Multiply on the left by the cofactor matrix, and we see

$$\begin{pmatrix} \det(1-\Phi) & & 0 \\ & \det(1-\Phi) & \\ 0 & & \ddots & \\ & & & \det(1-\Phi) \end{pmatrix} \begin{pmatrix} m_1 \\ \vdots \\ m_e \end{pmatrix}$$

$$= 0 .$$

In particular:

$$\underbrace{\{\det(I - (a_{ij}))\}}_{\substack{\shortparallel \\ \Delta.}} m_k = 0 \qquad \text{for } k = 1, \dots, e.$$

Then $\Delta \cdot M = 0$, since $M = \langle m_1, \dots, m_e \rangle$.

Now

$$\Delta = \begin{vmatrix} 1-a_{11} & -a_{12} & \cdots & -a_{1e} \\ -a_{21} & 1-a_{22} & \cdots & -a_{2e} \\ \vdots & & & \\ -a_{e1} & -a_{e2} & \cdots & 1-a_{ee} \end{vmatrix}$$

$$= 1 + x, \quad \text{where } x \in I.$$

<u>Proposition</u>: Let A, I, M be as above, with $M = IM$. Then $\exists\, x \in I$ such that $(1+x)M = 0$.
(Pf: $\Delta \cdot M = 0$, and $\Delta = 1+x$).

<u>Theorem (Nakayama's Lemma)</u>: Let A be a ring, I an ideal contained in $J(A)$, M a f.g. A-module such that $IM = M$. Then $M = 0$.

<u>Proof</u>:

We know $\exists\, x \in I$ s.t. $(1+x)M = 0$. From earlier observations, $1+x$ is a unit, since $x \in J(A)$.

Hence $M = 0$.

Corollary 1 Suppose $N \subseteq M$ is such that $M = N + IM$. Then $N = M$.
(Since this is a corollary to the Thm, the hypothesis of the Thm stand; in particular $I \subset J(A)$)

Proof: Apply the theorem to M/N, and note that $I\,(M/N) = \dfrac{IM+N}{N}$. //

Corollary 2: Suppose we have $x_1, \dots, x_p \in M$ such that their images $\bar{x}_1, \dots, \bar{x}_p \in M/IM$ generate M/IM as an A-module. Then $x_1, \dots, x_p$ generate M as an A-module.

Proof:
Let $N = \langle x_1, \dots, x_p \rangle \subseteq M$. Apply the previous corollary to N, for note that $M = IM + N$. //

The Spectrum of a commutative ring:

Let A be a ring. For an ideal I in A, define
$$V(I) = \{ \mathfrak{p} \in \mathrm{Spec}(A) \mid \mathfrak{p} \supseteq I \}.$$
Note that $V(I)$ is non-empty if and only if I is a proper ideal of A.

Easy to see that there is a bijection

$$\begin{aligned} V(I) &\longleftrightarrow \operatorname{Spec}(A/I) \\ \mathfrak{p} &\longmapsto \mathfrak{p}/I \end{aligned}$$

Facts:

$V(\bigcap_\alpha I_\alpha) = \bigcup V(I_\alpha)$ for finite intersections of I_α

$V(\sum_\alpha I_\alpha) = \bigcap V(I_\alpha)$, arbitrary sums of I_α.

$$V(A) = \phi$$

$$V(0) = \operatorname{Spec}(A).$$

Therefore we can define a topology on Spec (A) by decreeing that it closed sets are $V(I)$, I an ideal of A.

Remark: Suppose I is generated by a_α, $\alpha \in \Sigma$. Then $I = \sum_{\alpha \in \Sigma} \langle a_\alpha \rangle$. Hence, if you believe the above statements,

$$V(I) = \bigcap_{\alpha \in \Sigma} V(\langle a_\alpha \rangle).$$

Therefore open sets in Spec (A) are given by unions of sets of the form

$$D(a) := \operatorname{Spec}(A) \setminus V(\langle a \rangle)$$

Check (from comments made during the lecture on localizations) that
$$D(a) = \mathrm{Spec}(A_a)$$

Thus set $\mathcal{B} = \{D(a) \mid a \in A\}$ is a base for the topology on $\mathrm{Spec}(A)$, and
$$D(a) \cap D(b) = D(ab), \qquad a, b \in A.$$