We were proving Hilbert's Basis Theorem:
"If A is Noetherian and B is a finite type algebra over A, then B is Noetherian."
We'd reduced to the case $B = A[X]$.
Suppose $A[X]$ is not Noetherian. Then we can find an ascending

$$I_1 \subsetneq I_2 \subsetneq \dots \subsetneq I_n \subsetneq \dots$$

of ideals of $A[X]$ which doesn't become stationary.
Let $I = \bigcup_n I_n$. Let $f_1 \in I$ be an element s.t. $0 \neq f_1$ and $\deg(f_1) \leq \min\{\deg(f) \mid f \in I\}$.
We know that $\langle f_1 \rangle \subsetneq I$.
Now pick $f_2 \in I \setminus \langle f_1 \rangle$ such that $f_2 \neq 0$ and $\deg f_2 = \min\{\deg(f) \mid f \in I \setminus \langle f_1 \rangle\}$
Continue this way. If we've picked $f_1, \dots, f_k$, then pick f_{k+1} s.t. $f_{k+1} \neq 0$ and $f_{k+1} \in I \setminus \langle f_1, \dots, f_k \rangle$.
Let $n_i = \deg(f_i)$. Then

$$n_1 \leq n_2 \leq \dots \leq n_k \leq \dots$$

Let a_k = leading coefficient of f_k, $k \in \mathbb{N}$.
So $a_k \in A$.
Since A is Noetherian, the chain of ideals

$$\langle a_1\rangle \subset \langle a_1, a_2\rangle \subset \cdots \subset \langle a_1, \ldots, a_k\rangle \subset \cdots$$

in A must become stationary. Therefore, for some k we have

$$\langle a_1, \ldots, a_k\rangle = \langle a_1, \ldots, a_k, a_{k+1}\rangle$$

Therefore there exist $c_1, \ldots, c_k \in A$ s.t.

$$a_{k+1} = \sum_{j=1}^{k} c_j a_j$$

Let $$g = f_{k+1} - \sum_{j=1}^{k} c_j X^{n_{k+1} - n_j} f_j \in A[X]$$

Now $g \notin \langle f_1, \ldots, f_k\rangle$, for if it did, then $f_{k+1} \in \langle f_1, \ldots, f_k\rangle$. Therefore, by our choice of f_{k+1} and the defn of n_{k+1},

$$\deg(g) \geq n_{k+1}.$$

On the other hand, clearly $\deg(g) < n_{k+1}$ (check!). This is a contradiction. //

Remark: We've used the fact that if A is Noetherian, $A \overset{\phi}{\twoheadrightarrow} B$ a surjective ring homomorphism, then B is also Noetherian. Indeed, if J is an ideal of B, then $J = I/\ker\phi$ for a unique ideal I of A containing $\ker\phi$. By hyp., $I = \langle a_1, \ldots, a_k\rangle$, and hence $J = \langle \phi(a_1), \ldots, \phi(a_k)\rangle$.

Definition: Let A be a ring. An A-module M is said to be a Noetherian A-module if every submodule of M is finitely generated.

Remark: TFAE

1. M is Noetherian
2. Every chain of submodules of M
$$M_1 \subset M_2 \subset \cdots \subset M_n \subset \cdots$$
stabilizes (i.e. becomes stationary).
3. Every non-empty collection of submodules of M has a maximal element (w.r.t. inclusion).

Proof – old hat. //

Theorem (Hilbert's Basis Theorem for modules): Let A be a Noetherian ring and M a f.g. A-module. Then M is a Noetherian module.

Proof:

Note that the homomorphic image of a Noetherian module is Noetherian (proof exactly as in an earlier remark).

Since M is f.g., there is a surjective A-map

$$A^n \longrightarrow M, \quad e_i \longmapsto m_i$$

$$A^n \overset{\text{def.}}{=} \bigoplus_{i=1}^{n} A$$

standard basis of A^n

$m_1, \ldots, m_n$ generators of M.

Therefore, without loss of generality, we may assume $M = A^n$. We will so assume.

Since A is Noetherian, we are done if $n=1$. Assume A^{n-1} is Noetherian, for $n>1$.

We can identify A^{n-1} with the submodule

$$\{(0, a_2, a_3, \ldots, a_n) \mid a_2, \ldots, a_n \in A\}$$

of A^n.

Let

$$\pi : A^n \twoheadrightarrow A$$

be the projection to the first coordinate, i.e.,

$$\pi(a_1, \ldots, a_n) = a_1.$$

Now π is surjective.

Let $U \subseteq A^n$ be an A-submodule. Let $I = \pi(U) \subseteq A$. Then I is an ideal in A, and A being Noetherian, I is f.g.

Note $\ker(U \twoheadrightarrow I) = U \cap A^{n-1}$, where we are identifying A^{n-1} with a submodule of A^n as above. So we have a short exact sequence

$$0 \longrightarrow \underbrace{U \cap A^{n-1}}_{\text{f.g. by induction hypothesis}} \longrightarrow U \xrightarrow{\pi|_U} \underbrace{I}_{\text{f.g. since } A \text{ is Noetherian as a ring.}} \longrightarrow 0$$

It follows that U is f.g. //

Remark: Let $A \xrightarrow{f} B$ be a ring map, and $\mathfrak{q}$ a prime ideal in B. Then $f^{-1}(\mathfrak{q}) = \mathfrak{p}$ is a prime ideal in A. Indeed if $x, y \in A$ and $xy \in \mathfrak{p}$, then $f(xy) = f(x)f(y) \in \mathfrak{q}$, whence either $f(x)$ or $f(y)$ is in $\mathfrak{q}$, say $f(x)$ for definiteness. Then $x \in \mathfrak{p}$. (You may use this if you wish in Problem 6 of HW 1).

Localization at an element: Let A be a ring and $f \in A$ an element. Let $S = \{1, f, f^2, \ldots, f^n, \ldots\}$. Note that S is a multiplicative system. We write M_f for $S^{-1}M$, for $M \in \mathrm{Mod}_A$.

From problems in HW 1, the prime ideals of A_f obtained as follows: Let

$$\begin{aligned} A &\xrightarrow{L} A_f \\ x &\longmapsto x/1 \end{aligned}$$

be the localization map. Then $\mathfrak{Q} \subseteq A_f$ is a prime ideal of A_f if and only if $\mathfrak{p} = L^{-1}(\mathfrak{Q})$ is a prime ideal of A s.t. $f \notin \mathfrak{p}$.

The Nilradical and the Jacobson radical:

Let A be a ring. An element $a \in A$ is said to be nilpotent if $a^n = 0$ for some $n \geq 0$.

Proposition: The set of nilpotent elements of a ring A is an ideal of A.

Proof:

Let N be the set of nilpotents of A.
If $a \in A$, and $x \in N$, then clearly $ax \in N$.
Suppose $x, y \in N$. We have a non-negative integer n such that

$$x^n = y^n = 0.$$

Then

$$(x+y)^{2n} = \sum_{i=0}^{2n} \binom{2n}{i} x^i y^{2n-i} = 0$$

//

Definition: The set of nilpotent elements of a ring A is called the nilradical of A. It is often denoted $\sqrt{0}$. (Also often as $\text{rad}(A)$).

Theorem:

$$\sqrt{0} = \bigcap_{\substack{\mathfrak{p} \text{ a prime} \\ \text{ideal of } A}} \mathfrak{p}.$$

Proof:

Note the left side is contained in the right, for if x is nilpotent and $\mathfrak{p}$ a prime ideal, then $x^n = 0 \in \mathfrak{p}$ for some $n \geq 0$, whence $x \in \mathfrak{p}$.

It remains to show that if an element lies in every prime ideal it must be nilpotent.

We'll prove the contrapositive namely, if $a \in A$ is <u>not</u> nilpotent then there is a prime ideal $\mathfrak{p}$ of A which does not contain a.

Consider the localization A_a. We have shown (earlier) that every non-zero ring has a prime ideal.

<u>Claim</u>: $A_a \neq 0$.

<u>Pf</u>: If $A_a = 0$, then $\frac{a}{1} = 0$, i.e. for some $n \geq 0$, $a^n = 0$, i.e. a is nilpotent.

Therefore A_a has a prime ideal, say Q. Let $\mathfrak{p}$ be the inverse image of Q under the localization map $A \longrightarrow A_a$. We've seen that $\mathfrak{p}$ is a prime ideal of A which is disjoint from $S = \{a^n \mid n \geq 0\}$. It follows that $a \notin \mathfrak{p}$. //

(Have used Problem 6 from HW1).

Edit: Should be $\notin$ and not $\in$.