Jan 18, 2022   **Lecture 3**   Alg-II

- Noetherian rings & modules (Hilbert basis thm)
- Nilpotents, nilradical, Jacobson radical
- Determinant trick, Nakayama's lemma
- ?

Definition Let A be a ring, M an A-module, and $\{m_\alpha\}_{\alpha\in\Lambda}$ a collection of elements in M. The submodule of M generated by $\{m_\alpha\}$ is the smallest submodule of M containing every m_α. This is denoted $\langle m_\alpha \mid \alpha\in\Lambda\rangle$.

If $\{m_\alpha\} = \{m_1, \dots, m_k\}$ is a finite set, then we often write $\langle m_1, \dots, m_k\rangle$ for $\langle m_\alpha \mid \alpha\in\Lambda\rangle$.

Note: $\langle m_\alpha\rangle = \left\{ a_1 m_{\alpha_1} + \dots + a_k m_{\alpha_k} \;\middle|\; a_i \in A,\ i=1,\dots,k\right\}$, i.e. $\langle m_\alpha\rangle$ is the set of finite linear combinations (with scalars from A) of elements in $\{m_\alpha \mid \alpha\in\Lambda\}$.

An A-module M is said to be finitely generated if there exist a finite number of elements $m_1, \dots, m_k \in M$ s.t. $M = \langle m_1, \dots, m_k\rangle$.

Since an ideal of A is the same as an A-submodule of A, it makes to talk about finitely generated ideals in A.

2   There are three possibilities for denoting a principal ideal; $(a) = \langle a\rangle = aA$.

Theorem: Let A be a ring. The following are equivalent:

(a) Every ideal of A is finitely generated

(b) The ascending chain condition holds: Every ascending chain of ideal in A

$$I_1 \subset I_2 \subset I_3 \subset \dots \subset I_n \subset \dots$$

becomes stationary.

(c) The maximal condition for ideals holds: If Σ is a non-empty set of ideals in A, then Σ contains a maximal element (w.r.t. inclusion).

Proof:

(a) $\Rightarrow$ (b) Let

$$I_1 \subset I_2 \subset I_3 \subset \dots \subset I_n \subset \dots$$

be an ascending chain of ideals. Then

$$I = \bigcup_n I_n$$

is an ideal. By hypothesis, $I = \langle a_1, \dots, a_k \rangle$ for some $a_1, \dots, a_k \in A$. Each $a_i \in I_{j_i}$ for some j_i. Pick n s.t. $n \geq j_1, \dots, j_k$. Then $a_1, \dots, a_k \in I_{n+m}$, for all $m \geq 0$. It follows that

$$I = \langle a_1, \dots, a_k \rangle \subseteq I_{n+m} \subseteq I \quad \forall \ m \geq 0.$$

Hence $I = I_n = I_{n+1} = \dots = I_{n+m} = \dots$, i.e. the chain of ideals $\dots \subset I_j \subset I_{j+1} \subset \dots$ becomes stationary.

(b) $\Rightarrow$ (c). Suppose Σ is a non-empty collection of ideals in A. Suppose Σ does not have a maximal element. Pick $I_1 \in \Sigma$. Then there exists $I_2 \in \Sigma$ s.t. $I_1 \subsetneq I_2$. Suppose we have picked $I_1 \subsetneq I_2 \subsetneq \dots \subsetneq I_k$ with each $I_j \in \Sigma$. Since Σ does not have a max'l element, there exists $I_{k+1} \in \Sigma$ s.t. $I_k \subsetneq I_{k+1}$. We have therefore found an ascending chain of ideals which is not stationary, contradicting (b).

(c) $\Rightarrow$ (a) Let I be an ideal which is not finitely generated. Pick $a_1 \in I$. Now $\langle a_1 \rangle \subsetneq I$, and hence $\exists\ a_2 \in I \setminus \langle a_1 \rangle$. Suppose we have picked $a_1, \dots, a_k \in I$ s.t.

$$\langle a_1 \rangle \subsetneq \langle a_1, a_2 \rangle \subsetneq \dots \subsetneq \langle a_1, \dots, a_k \rangle$$

Since I is not f.g, $\langle a_1, \dots, a_k \rangle \subsetneq I$, and hence we can find $a_{k+1} \in I \setminus \langle a_1, \dots, a_k \rangle$. Then $\langle a_1, \dots, a_k \rangle \subsetneq \langle a_1, \dots, a_k, a_{k+1} \rangle$. Inductively we have constructed a collection of ideals $\{ \langle a_1, \dots, a_k \rangle \mid k = 1, 2, \dots \}$ which has no max'l element. //

Definition: A ring A is said to be noetherian if it satisfies any of the equivalent conditions of the above theorem.

Definition: Let $f : A \longrightarrow B$ be a ring homomorphism. We say B is of finite-type over A (or finitely generated as an algebra over A) if there exist a finite number of elements $b_1, \ldots, b_k \in B$ such that every member b of B can be written as finite sum of the form

$$b = \sum_{\mu_i \geq 0} f(a_{\mu_1 \ldots \mu_k})\, b_1^{\mu_1} \cdots b_k^{\mu_k} \quad , \quad a_{\mu_1 \ldots \mu_k} \in A.$$

Short exact sequences:

A sequence of A-modules

$$M^\bullet : \quad \cdots \longrightarrow M^i \xrightarrow{d^i} M^{i+1} \xrightarrow{d^{i+1}} \cdots$$

is called a complex (or a cohomology complex) if $d^{i+1} \circ d^i = 0 \quad \forall i$.

$M^\bullet$ is said to be exact at i if

$$\ker(d^{i+1}) = \operatorname{im}(d^i).$$

$M^\bullet$ is said to be exact if it is exact at every i.

A short exact sequence is a complex of the form

(*) $$0 \longrightarrow M' \xrightarrow{\phi} M \xrightarrow{\psi} M'' \longrightarrow 0$$

which is exact, i.e.,

$$\left.\begin{array}{l} \bullet \ \ker \phi = 0 \\ \bullet \ \operatorname{im} \phi = \ker \psi \\ \bullet \ \operatorname{im} \psi = M'' \end{array}\right\}$$

Let (*) be exact.

A little thought shows that the following diagram commutes, with downward arrows being isomorphism

$$
\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "\phi"] \arrow[d, "\wr"] & M \arrow[r, "\psi"] \arrow[d, equal] & M'' \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & K \arrow[r] & M \arrow[r] & M/K \arrow[r] & 0
\end{tikzcd}
$$

$K = \ker\psi$

natural inclusion

natural surjection $(m \longmapsto m+K)$

Lemma: Let A be a ring and

$$0 \longrightarrow M' \xrightarrow{\phi} M \xrightarrow{\psi} M'' \longrightarrow 0$$

a short exact sequence of A-modules. If M' and M'' are f.g. as A-modules then M is f.g as an A-module.

Proof:

We will identify M' with $\ker(M \to M'')$ and M'' with M/M' (see Remark above).

Suppose $M'' = \langle y_1, \dots, y_k \rangle$, $y_i \in M''$ and
$M' = \langle x_1, \dots, x_\ell \rangle$, $x_i \in M'$.

Since $M \longrightarrow M''$ is surjective, we can (and will) pick pre-images $m_1, \dots, m_k \in M$ of $y_1, \dots, y_k$ with $m_i \longmapsto y_i$, $i = 1, \dots, k$.

Let $m \in M$. Let $y = \varphi(m)$. Then $\exists$ $a_1, \dots, a_k \in A$ s.t.

$$y = a_1 y_1 + \dots + a_k y_k.$$

Now consider

$$z = m - (a_1 m_1 + \dots + a_k m_k) \qquad \text{———} (*)$$

Clearly

$$\varphi(z) = \varphi(m) - \sum_{i=1}^{k} a_i \varphi(m_i)$$
$$= \varphi(m) - \varphi(m)$$
$$= 0$$

So $z \in \ker(\varphi) = M'$. Since $M' = \langle x_1, \dots, x_\ell \rangle$,

$$z = \sum_{j=1}^{\ell} \alpha_j x_j, \quad \alpha_j \in A. \qquad \text{———} (**)$$

From $(*)$ and $(**)$, get

$$m = \sum_{j=1}^{\ell} \alpha_j x_j + \sum_{i=1}^{k} a_i m_i$$

It follows that $M = \langle x_1, \dots, x_\ell, m_1, \dots, m_k \rangle$. //

Theorem (Hilbert's Basis Theorem): Let A be a noetherian ring and B an A-algebra of finite type. Then B is also noetherian.

Proof:

Since B is of finite type, there is a surjective ring homomorphism

$$A[X_1, \dots, X_n] \twoheadrightarrow B$$

where $A[X_1, \dots, X_n]$ is the poly. ring in n-variables over A.

In greater detail, suppose $b_1, \ldots, b_n \in B$ are algebra generators for B over A. Consider the map

$$A[X_1, \ldots, X_n] \longrightarrow B$$

given by

$$p \longmapsto p(b_1, \ldots, b_n)$$

Since every element of B is of the form $\sum_i a_{\mu_1 \ldots \mu_n} b_1^{\mu_1} \cdots b_n^{\mu_n}$, this map is surjective.

Clearly it is therefore enough to show that $A[X_1, \ldots, X_n]$ is noetherian.

By recursion (or induction) it is enough to prove $A[X]$ is noetherian.