

Today's topics:

- Local rings
- Multiplicative systems
- Modules
- Localisation / Localization
- Noetherian rings (?)

Local rings: A ring A is said to be a local ring if it has exactly one maximal ideal.

Notations: 1. If \mathfrak{m} is the ! maximal ideal of a local ring, we often write (A, \mathfrak{m}) is a local ring.
2. A local ring $\longleftrightarrow \mathfrak{m}_A = ! \text{ max'l ideal of } A$.

Lemma: Let A be a ^{non-zero} ring. Then the set of maximal ideals of A is non-empty.

Proof: Let Σ be the collection of proper ideals in A (I is a proper ideal of A if I is an ideal and $I \neq A$). $\Sigma \neq \emptyset$ since $0 \in \Sigma$. Order Σ by inclusion. It is clear that we have chain of ideals $I_1 \subset \dots \subset I_\beta$ in Σ , then $I = \bigcup_{\alpha} I_\alpha \in \Sigma$.

By Zorn's lemma we're done //

Multiplicative systems: Let A be a ring. A subset S of A is called a multiplicative system if it is closed under multiplication and $1 \in S$.

Modules: Let A be a ring. An A -module or a module over A (or simply a module if the context is clear) is an abelian group M together with a "scalar multiplication map"

$$\begin{array}{ccc} A \times M & \longrightarrow & M \\ (a, m) & \longmapsto & am \quad \text{or } a \cdot m \end{array}$$

such that

- (i) $(a_1 a_2) \cdot m = a_1 (a_2 m)$
 - (ii) $(a_1 + a_2) m = a_1 m + a_2 m$
 - (iii) $a (m_1 + m_2) = a m_1 + a m_2$
 - (iv) $1 \cdot m = m \quad \forall m \in M.$
- $(a_1, a_2 \in A, m \in M)$
 $(a \in A, m_1, m_2 \in M)$

Remarks: 1. The ring A is clearly an A -module with the obvious notion of scalar product. Every ideal of A is an A -module.

2. Let M be an A -module. There is an obvious notion of a sub-module N of M , namely N is an additive subgroup of M , and is closed under scalar multiplication on M . In other words

N is an A -module with scalar multiplication inherited from M .

3. An ideal is the same as a submodule of the A -module A .

Localization: Let A be a ring, M an A -module and S a multiplicative system in A . The localization of M at S , $S^{-1}M$ or M_S , is constructed as follows:

Define an equivalence relation \sim on $S \times M$ by

$$(s, m) \sim (s', m')$$

if $\exists s'' \in S$ such that

$$s''(s'm - sm') = 0.$$

Write $\frac{m}{s}$ for the equivalence of (s, m) , and M_S (or $S^{-1}M$) for $S \times M / \sim$.

Can make M_S into an A -module as follows:

$$\bullet \quad \frac{m_1}{s_1} + \frac{m_2}{s_2} = \frac{s_2 m_1 + s_1 m_2}{s_1 s_2}$$

$$\bullet \quad s' \left(\frac{m}{s} \right) = \frac{s' m}{s}.$$

It is easy to see that this gives us a module structure on M_S .

Qn: What if $0 \in S$. What is $\frac{m}{s}$? In fact,

in this case $M_S = 0$, since

$$(s, m) \sim (s', m')$$

$\forall (s, m), (s', m') \in S \times M$. To see this, note that $0(s'm - sm') = 0$

How do multiplicative systems arise?

Let \mathfrak{p} be a prime ideal of a ring A .

Let $S = A - \mathfrak{p}$. Then S is a multiplicative system (and $0 \notin S$, yay!).

Notation: $A_{\mathfrak{p}} := S^{-1}A$. ($= A_S$)

\uparrow
The localization of A at \mathfrak{p} .

Check: • $A_{\mathfrak{p}}$ is a local ring.

• More generally, check that if S is an arbitrary multiplicative system, then

$S^{-1}A$ is a ring, with product:

$$\frac{a}{s} \cdot \frac{a'}{s'} = \frac{aa'}{ss'}$$

Universal property of localization:

If S is a multiplicative system in a ring A , and $M \in \text{Mod}_A$ \leftarrow the category of A -modules, then

we have a natural map

$$M \longrightarrow S^{-1}M$$

namely $m \longmapsto \frac{m}{1}$.

This is an A -module homomorphism: i.e. it respects addition and scalar multiplication.

Moreover, the map $A \longrightarrow S^{-1}A$ (set $M=A$ in the above discussion) is a ring homomorphism.

e.g. $ab \mapsto \frac{ab}{1} = \frac{a}{1} \cdot \frac{b}{1}$

Finally, $S^{-1}M$ is an $S^{-1}A$ -module via the operation

$$\frac{a}{s} \left(\frac{m}{t} \right) := \frac{am}{st}.$$

$$\begin{array}{c} * \quad \text{---} \quad * \end{array}$$

$$A \longrightarrow S^{-1}A$$

$$M \longrightarrow S^{-1}M$$

$$\begin{array}{c} * \quad \text{---} \quad * \end{array}$$

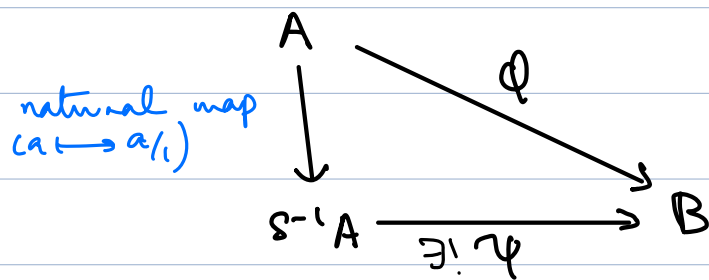
Proposition: Let A be a ring, $S \subseteq A$ a multiplicative, and M an A -module.

(a) Let $\phi: A \longrightarrow B$ be a ring homomorphism such that $\phi(s)$ is a unit in B for all $s \in S$.

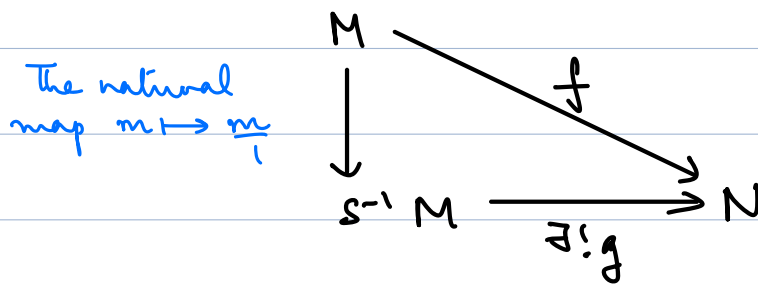
Then there exists a unique ring homomorphism

$$\psi: S^{-1}A \longrightarrow B$$

such that the following diagram commutes:



(b) If $f: M \rightarrow N$ is an A -module homomorphism, and if for each $s \in S$, the map $\mu_s: N \rightarrow N$ given by $x \mapsto sx$, $x \in N$, $s \in S$, is bijective, then $\exists!$ map $g: S^{-1}M \rightarrow N$ such that TFDC



Proof:

Part (a) : Define $\psi: S^{-1}A \rightarrow B$ by the formula

$$\psi\left(\frac{a}{s}\right) = \phi(s)^{-1} \phi(a), \quad s \in S, a \in A.$$

(Recall, $\phi(1)$ is a unit in B by hypothesis.)

Suppose $\frac{a}{s} = \frac{a'}{s'}$. Then $\exists t \in S$ such that

$$t(s'a - sa') = 0.$$

Apply ϕ to this "equation". Get

$$\phi(t) (\phi(s') \phi(a) - \phi(s) \phi(a')) = 0$$

Now, $\phi(1)$ is a unit in B . Multiplying the above by $\phi(1)^{-1}$ we get

$$\phi(1') \phi(a) = \phi(1) \phi(a')$$

$$\text{i.e. } \phi(1')^{-1} \phi(a) = \phi(1')^{-1} \phi(a')$$

thus establishing well-definedness of ψ .

Will leave it to you to check that ψ is a ring homomorphism (e.g. $\psi(\frac{a}{s} \cdot \frac{a'}{s'}) = \phi(1's')^{-1} \phi(aa')$
 $= \phi(1')^{-1} \phi(1s)^{-1} \phi(a) \phi(a')$
 $= \phi(1')^{-1} \phi(a) \phi(1s')^{-1} \phi(a')$
 $= \psi(\frac{a}{s}) \cdot \psi(\frac{a'}{s'}).$)

Part (b) is similar. //

Theorem: Let A be a ring. The following are equivalent:

(a) Every ideal of A is finitely generated

(b) The ascending chain condition holds: Every ascending chain of ideal in A

$$I_1 \subset I_2 \subset I_3 \subset \dots \subset I_n \subset \dots$$

becomes stationary.

(c) The maximal condition for ideals holds: If Σ is a non-empty set of ideals in A , then Σ contains a maximal element (w.r.t. inclusion).

Definition: A ring is called Noetherian ^{← noetherian} if it satisfies any of the equivalent conditions of the above theorem.