Today's topics:

- Local rings
- Multiplicative systems
- Modules
- Localisation / Localization
- Noetherian rings (?)

Local rings: A ring A is said to be a local ring if it has exactly one maximal ideal.

Notations: 1. If $\mathfrak{m}$ is the ! maximal ideal a local ring, we often write $(A, \mathfrak{m})$ is a local ring.
2. A local ring $\longleftrightarrow$ $\mathfrak{m}_A$ = ! max'l ideal of A.

Lemma: Let A be a non-zero ring. Then the set of maximal ideals of A is non-empty.

Proof: Let Σ be the collection of proper ideals in A (I is a proper ideal of A if I is an ideal and $I \neq A$). $\Sigma \neq \phi$ since $0 \in \Sigma$. Order Σ by inclusion. It is clear that we have chain of ideals $I_\alpha \subset \dots \subset I_\beta$ in Σ, then

$$I = \bigcup_\alpha I_\alpha \in \Sigma.$$

By Zorn's lemma we're done //

Multiplicative systems: Let A be a ring. A subset S of A is called a multiplicative system if it is closed under multiplication and $1 \in S$.

Modules: Let A be a ring. An A-module or a module over A (or simply a module if the context is clear) is an abelian group M together with a "scalar multiplication map"

$$\begin{aligned} A \times M &\longrightarrow M \\ (a, m) &\longmapsto am \quad \text{or } a\cdot m \end{aligned}$$

such that

$$\left.\begin{aligned} &\text{(i)} \quad (a_1 a_2)\cdot m = a_1(a_2 m) \\ &\text{(ii)} \quad (a_1 + a_2) m = a_1 m + a_2 m \end{aligned}\right\} (a_1, a_2 \in A, m \in M)$$

$$\text{(iii)} \quad a(m_1 + m_2) = am_1 + am_2 \quad (a \in A, m_1, m_2 \in M)$$

$$\text{(iv)} \quad 1\cdot m = m \quad \forall\, m \in M.$$

Remarks: 1. The ring A is clearly an A-module with the obvious notion of scalar product. Every ideal of A is an A-module.

2. Let M be an A-module. There is an obvious notion of a sub-module N of M, namely N is an additive subgroup of M, and is closed under scalar multiplication on M. In other words

N is an A-module with scalar multiplication inherited from M.

3. An ideal is the same as a submodule of the A-module A.

Localization: Let A be a ring, M an A-module and S a multiplicative system in A. The localization of M at S, $S^{-1}M$ or M_S, is constructed as follows:

Define an equivalence relation $\sim$ on $S \times M$ by

$$(s, m) \sim (s', m')$$

if $\exists \; s'' \in S$ such that

$$s''(s'm - sm') = 0.$$

Write $\frac{m}{s}$ for the equivalence of (s,m), and M_S (or $S^{-1}M$) for $S \times M/\sim$.

Can make M_S into an A-module as follows:

- $$\frac{m_1}{s_1} + \frac{m_2}{s_2} = \frac{s_2 m_1 + s_1 m_2}{s_1 s_2}$$

- $$s'\left(\frac{m}{s}\right) = \frac{s'm}{s}.$$

It is easy to see that this gives us a module structure on M_S.

Qn: What if $0 \in S$. What is $\frac{m}{s}$? In fact, in this case $M_S = 0$, since

$$(s,m) \sim (s',m')$$

$\forall$ $(s,m), (s',m') \in S \times M$. To see this, note that $0(s'm - sm') = 0$

How do multiplicative systems arise?

Let $\mathfrak{p}$ be a prime ideal of a ring A.
Let $S = A - \mathfrak{p}$. Then S is a multiplicative system (and $0 \notin S$, yay!).

<u>Notation</u>: $A_{\mathfrak{p}} := S^{-1}A$. $(= A_S)$

The localization of A at $\mathfrak{p}$.

<u>Check</u>: • $A_{\mathfrak{p}}$ is a local ring.

• More generally, check that if S is an arbitrary multiplicative system, then $S^{-1}A$ is a ring, with product:

$$\frac{a}{s} \cdot \frac{a'}{s'} = \frac{aa'}{ss'}.$$

Universal property of localization:

If S is a multiplicative system in a ring A, and $M \in \mathrm{Mod}_A$ ← the category of A-modules, then we have a natural map

$$M \longrightarrow S^{-1}M$$

namely $m \longmapsto \frac{m}{1}$.

This is an A-module homomorphism: i.e. it respects addition and scalar multiplication.

Moreover, the map $A \longrightarrow S^{-1}A$ (set $M=A$ in the above discussion) is a ring homomorphism.

e.g. $ab \longmapsto \frac{ab}{1} = \frac{a}{1} \cdot \frac{b}{1}$

Finally, $S^{-1}M$ is an $S^{-1}A$-module via the operation

$$\frac{a}{s}\left(\frac{m}{t}\right) := \frac{am}{st}.$$

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$$A \longrightarrow S^{-1}A$$

$$M \longrightarrow S^{-1}M$$

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Proposition: Let A be a ring, $S \subseteq A$ a multiplicative, and M an A-module.

(a) Let $\phi : A \longrightarrow B$ be a ring homomorphism such that $\phi(s)$ is a unit in B for all $s \in S$. Then there exists a unique ring homomorphism

$$\psi : S^{-1}A \longrightarrow B$$

such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[d, "\text{natural map } (a \mapsto a/1)"'] \arrow[dr, "\phi"] & \\ S^{-1}A \arrow[r, "\exists!\, \psi"'] & B \end{tikzcd}$$

(b) If $f: M \longrightarrow N$ is an A-module homomorphism, and if for each $s \in S$, the map $\mu_s : N \longrightarrow N$ given by $x \longmapsto sx$, $x \in N$, $s \in S$, is bijective, then $\exists!$ map $g: S^{-1}M \longrightarrow N$ such that TFDC

$$\begin{tikzcd} M \arrow[d, "\text{The natural map } m \mapsto \frac{m}{1}"'] \arrow[dr, "f"] & \\ S^{-1}M \arrow[r, "\exists!\, g"'] & N \end{tikzcd}$$

Proof:

Part (a) : Define $\psi : S^{-1}A \longrightarrow B$ by the formula

$$\psi\left(\frac{a}{s}\right) = \phi(s)^{-1}\phi(a). \quad , s \in S, a \in A.$$

(Recall, $\phi(s)$ is a unit in B by hypothesis.)

Suppose $\frac{a}{s} = \frac{a'}{s'}$. Then $\exists$ $t \in S$ such that

$$t(s'a - sa') = 0.$$

Apply ϕ to this "equation". Get

$$\phi(t)\left(\phi(s')\phi(a) - \phi(s)\phi(a')\right) = 0$$

Now, $\phi(t)$ is a unit in B. Multiplying the above by $\phi(t)^{-1}$ we get

$$\phi(s')\,\phi(a) = \phi(s)\,\phi(a')$$

i.e. $$\phi(s)^{-1}\phi(a) = \phi(s')^{-1}\phi(a')$$

thus establishing well-definedness of ψ.

Will leave it to you to check that ψ is a ring homomorphism (e.g. $\psi\left(\frac{a}{s}\cdot\frac{a'}{s'}\right) = \phi(ss')^{-1}\phi(aa')$
$= \phi(s)^{-1}\phi(s')^{-1}\phi(a)\phi(a')$
$= \phi(s)^{-1}\phi(a)\ \phi(s')^{-1}\phi(a')$
$= \psi\left(\frac{a}{s}\right)\cdot\psi\left(\frac{a'}{s'}\right).$)

Part (b) is similar. //

Theorem: Let A be a ring. The following are equivalent:

(a) Every ideal of A is finitely generated

(b) The ascending chain condition holds: Every ascending chain of ideal in A
$$I_1 \subset I_2 \subset I_2 \subset \cdots \subset I_n \subset \cdots$$
becomes stationary.

(c) The maximal condition for ideals holds: If Σ is a non-empty set of ideals in A, then Σ contains a maximal element (w.r.t. inclusion).

Definition: A ring is called Noetherian (noetherian) if it satisfies any of the equivalent conditions of the above theorems.