LECTURE 18

Date of Lecture: March 17, 2022

We fix a ring R throughout the lecture, and all modules appearing are R-modules. Complexes will be complexes of R-modules.

1. Mapping cones

1.1. **Conventions.** Let $M = \bigoplus_{j=1}^{e} M_j$ and $N = \bigoplus_{i=1}^{d} N_i$ be R-modules. We will represent an R-map $\varphi\colon M \to N$ in a matrix form

$$\varphi = \begin{bmatrix} \varphi_{11} & \dots & \varphi_{1e} \\ \vdots & \ddots & \vdots \\ \varphi_{d1} & \dots & \varphi_{de} \end{bmatrix}$$

where $\varphi_{ij} \in \mathrm{Hom}_R(M_j,\, N_i)$, $1 \le i \le d$ and $1 \le j \le e$. If we write elements of M (respectively N) as column vectors, with the j^{th} (repectively i^{th}) entry from M_j (respectively N_i), then one checks easily that for $m \in M$ whose j^{th} component is m_j,

$$\varphi(m) = \begin{bmatrix} \varphi_{11} & \dots & \varphi_{1e} \\ \vdots & \ddots & \vdots \\ \varphi_{d1} & \dots & \varphi_{de} \end{bmatrix} \begin{bmatrix} m_1 \\ \vdots \\ m_e \end{bmatrix}$$

In these matters, it is useful to make a distinction between the row matrix $\begin{bmatrix} m_1 & \dots & m_e \end{bmatrix}$ and the d-tuple $(m_1, \dots, m_e)$. We identify the latter with the column vector $\begin{bmatrix} m_1 \\ \vdots \\ m_e \end{bmatrix}$. Thus

$$(m_1, \dots, m_e) \neq \begin{bmatrix} m_1 & \dots & m_e \end{bmatrix}. \tag{1.1.1}$$

Each side is the transpose of the other. In particular, elements of M can be written as d-tuples which is typographically more convenient.

It is easy to see that if $L = \bigoplus_{k=1}^{f} L_k$ and $\psi\colon L \to M$ is second map, and $\psi = [\psi_{jk}]$ the corresponding matrix representation, then $\varphi \circ \psi$ is represented by the matrix multiplication $[\varphi_{ij}][\psi_{jk}]$.

1.2. **Translations.** Let $A^\bullet$ be a complex and n an integer. The *translation of $A^\bullet$ by n units* is the complex $A^\bullet[n]$, where

$$(A^\bullet[n])^i = A^{n+i} \quad \text{and} \quad d^i_{A^\bullet[n]} = (-1)^n d^{n+i}_{A^\bullet}. \tag{1.2.1}$$

If $n > 0$, $A^\bullet[n]$ is essentially the translation of $A^\bullet$ by n units to the left. Note that

$$\mathrm{H}^i(A^\bullet[n]) = \mathrm{H}^{n+i}(A^\bullet). \tag{1.2.2}$$

1.3. **Mapping cones.** Let $\varphi\colon A^\bullet \to B^\bullet$ be a map of complexes. Define the *mapping cone of* φ to be the complex $C^\bullet_\varphi$ whose n^{th} graded piece is

$$C^n_\varphi = B^n \oplus A^{n+1} \tag{1.3.1}$$

and whose differentials $d^n_{C_\varphi}\colon C^n_\varphi \to C^{n+1}_\varphi$ are gives by the formula

$$d^n_{C_\varphi} = \begin{bmatrix} d^n_B & \varphi^{n+1} \\ 0 & -d^{n+1}_A \end{bmatrix} \tag{1.3.2}$$

Now $\begin{bmatrix} d^{n+1}_B & \varphi^{n+2} \\ 0 & -d^{n+2}_A \end{bmatrix}\begin{bmatrix} d^n_B & \varphi^{n+1} \\ 0 & -d^{n+1}_A \end{bmatrix} = \begin{bmatrix} d^{n+1}_B d^n_B & d^{n+1}_B\varphi^{n+1} - \varphi^{n+1}d^{n+1}_A \\ 0 & d^{n+2}_A d^{n+1}_A \end{bmatrix} = 0$, where we are using the fact that φ is a cochain map to conclude that $d^{n+1}_B\varphi^{n+1} - \varphi^{n+2}d^{n+1}_A = 0$. In other words $(C^\bullet_\varphi, d_{C^\bullet_\varphi})$ is indeed a complex.

There is an obvious short exact sequence of graded R-modules

$$0 \longrightarrow B^\bullet \xrightarrow{\ i\ } C^\bullet_\varphi \xrightarrow{\ \pi\ } A^\bullet[1] \longrightarrow 0 \tag{1.3.3}$$

where $i^n\colon B^n \to C^n_\varphi = B^n \oplus A^{n+1}$ is the canonical inclusion, and $\pi^n\colon B^n \to A^{n+1}$ is the canonical projection $B^n \oplus A^{n+1} \to A^{n+1}$. In matrix notation

$$i^n = \begin{bmatrix} \mathbf{1} \\ 0 \end{bmatrix} \quad \text{and} \quad \pi^n = \begin{bmatrix} 0 & \mathbf{1} \end{bmatrix}.$$

Thus, at the n^{th} level (1.3.3) is standard split exact sequence

$$0 \longrightarrow B^n \longrightarrow B^n \oplus A^{n+1} \longrightarrow A^{n+1} \longrightarrow 0$$

We claim that (1.3.3) is a short exact sequence of complexes (upgrading it from a mere short exact sequence of graded modules). This follows from elementary "matrix multiplication", in other words from the following readily verified identities:

$$\begin{bmatrix} \mathbf{1} \\ 0 \end{bmatrix}\left[d^n_B\right] = \begin{bmatrix} d^n_B & \varphi^{n+1} \\ 0 & -d^{n+1}_A \end{bmatrix}\begin{bmatrix} \mathbf{1} \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & \mathbf{1} \end{bmatrix}\begin{bmatrix} d^n_B & \varphi^{n+1} \\ 0 & -d^{n+1}_A \end{bmatrix} = \left[-d^{n+1}_A\right]\begin{bmatrix} 0 & \mathbf{1} \end{bmatrix}$$

Before we give in to our natural instinct and write out the long exact sequence associated to (1.3.3), let us work out the connecting map $\delta\colon \mathrm{H}^n(A^\bullet[c]) \to \mathrm{H}^{n+1}(B^\bullet)$. To that end, suppose $a \in Z^n(A^\bullet[1]) = Z^{n+1}(A^\bullet)$. A pre-image in C^n_φ is $(0, a)$ (keep in mind our conventions, see especially (1.1.1) and the conventions mentioned around that relation). Now $d_{C_\varphi}(0, a) = (\varphi(a), -d_A(a)) = (\varphi(a)), 0) = i(\varphi(a))$. Hence $\delta[a] = [\varphi[a]$, In other words $\delta = \varphi_*$. In greater detail, the following diagram commutes

$$\begin{array}{ccc} \mathrm{H}^n(A^\bullet[1]) & \xrightarrow{\ \delta\ } & \mathrm{H}^{n+1}(B^\bullet) \\ \| & & \| \\ \mathrm{H}^{n+1}(A^\bullet) & \xrightarrow[\ \varphi_*\ ]{} & \mathrm{H}^{n+1}(B^\bullet) \end{array} \tag{1.3.4}$$

The long exact sequence associated with (1.3.3) then is:

$$\ldots \longrightarrow \mathrm{H}^n(A^\bullet) \xrightarrow{\varphi_*} \mathrm{H}^n(B^\bullet) \xrightarrow{i_*} \mathrm{H}^n(C^\bullet_\varphi) \xrightarrow{\pi_*} \mathrm{H}^{n+1}(A^\bullet) \xrightarrow{\varphi_*} \ldots \tag{1.3.5}$$

1.4. **Quasi-isomorphisms.** A map of complexes $\varphi\colon A^\bullet \to B^\bullet$ is said to be a *quasi-isomorphism* if $\varphi_*\colon \mathrm{H}^n(A^\bullet) \to \mathrm{H}^n(B^\bullet)$ is an isomorphism for every $n \in \mathbf{Z}$.

Here is the main theorem

Theorem 1.4.1. *A map of complexes $\varphi\colon A^\bullet \to B^\bullet$ is a quasi-isomorphism if and only if the mapping cone $C^\bullet_\varphi$ is exact.*

Proof. Consider the induced exact sequence (1.3.5). If $C^\bullet_\varphi$ is exact, then $\mathrm{H}^n(C^\bullet_\varphi) = 0$ for all n, whence $0 \to \mathrm{H}^n(A^\bullet) \xrightarrow{\varphi_*} \mathrm{H}^n(B^\bullet) \to 0$ is exact for all n, which means $\varphi_*\colon \mathrm{H}^n(A^\bullet) \to \mathrm{H}^n(B^\bullet)$ is an isomorphism for every n.

For the converse, suppose φ is a quasi-isomorphism. From the exact sequence (1.3.5) one deduces that $\ker i_* = \mathrm{im}(\varphi_*) = \mathrm{H}^n(B^\bullet)$, whence $i_*\colon \mathrm{H}^n(B^\bullet) \to \mathrm{H}^n(C^\bullet_\varphi)$ is the zero map. Since $\mathrm{im}(\pi_*) = \ker \varphi_* = 0$, we see that $\pi_*\colon \mathrm{H}^n(C^\bullet_\varphi) \to \mathrm{H}^{n+1}(A^\bullet)$ is the zero map. Since the incoming and the outgoing maps at $\mathrm{H}^n(C^\bullet_\varphi)$ are zero in the exact sequence (1.3.5), $\mathrm{H}^n(C^\bullet_\varphi) = 0$. In other words $C^\bullet_\varphi$ is exact. ☐

References

[I] B. Iversen, *Cohomology of Sheaves*, Universitext, Springer-Verlag, Berlin, 1986.