LECTURE 17

Date of Lecture: March 15, 2022

1. Short exact sequence of complexes

1.1. **The connecting homomorphism.** We defined the notion of an exactness on pp.3–4 of Lecture 10. It is clear that a complex $C^\bullet$ is exact at the n^{th} place if and only if $\mathrm{H}^n(C^\bullet) = 0$ and it is exact if and if $\mathrm{H}^n(C^\bullet) = 0$ for all $n \in \mathbf{Z}$.

For a cochain map $\varphi\colon P^\bullet \to Q^\bullet$, we will write φ_* for $\mathrm{H}^n(\varphi)$, and suppress the superscript n.

Let

$$0 \longrightarrow P^\bullet \xrightarrow{\varphi} Q^\bullet \xrightarrow{\psi} R^\bullet \longrightarrow 0 \tag{1.1.1}$$

be a *short exact sequence* of complexes, i.e. in the following commutative diagram (which has an infinite number of rows), every row is exact.

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
0 & \longrightarrow & P^{n+1} & \xrightarrow{\varphi^{n+1}} & Q^{n+1} & \xrightarrow{\psi^{n+1}} & R^{n+1} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
0 & \longrightarrow & P^{n} & \xrightarrow{\varphi^{n}} & Q^{n} & \xrightarrow{\psi^{n}} & R^{n} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
0 & \longrightarrow & P^{n-1} & \xrightarrow[\varphi^{n-1}]{} & Q^{n-1} & \xrightarrow[\psi^{n-1}]{} & R^{n-1} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array} \tag{1.1.2}$$

Given a short exact sequence of complexes, there are the so called "connecting maps" from $\mathrm{H}^n(R^\bullet)$ to $\mathrm{H}^{n+1}(P^\bullet)$. To define them we first define a map from $Z^n(R^\bullet)$ to $\mathrm{H}^{n+1}(P^\bullet)$, and this map "descends" to $\mathrm{H}^n(R^\bullet)$. In defining this map it will be useful to look at the commutative diagram (1.1.2) to see where the elements which come up in the "diagram chase" live.

To reduce notational baggage, we will drop superscripts for maps (e.g. we will write φ for φ^j). Let $r \in Z^n(R^\bullet)$, and pick a pre-image $q \in Q^n$ of r (under ψ). Then $\psi(dq) = dr = 0$. This means there is a unique element $p \in P^{n+1}$ such that $\varphi(p) = dq$. Moreover, since $\varphi(dp) = d(dq) = 0$, whence $dp = 0$ (φ being injective). In other words $p \in Z^{n+1}(P^\bullet)$. Let $[p] \in \mathrm{H}^{n+1}(P^\bullet)$ be the cohomology class determined by p (or, in plain terms, let $[p] = p + B^{n+1}(P^\bullet)$). The situation is perhaps best portrayed in the following schematic way:

$$
\begin{array}{ccc}
p & \longmapsto & dq \\
 & & \uparrow \\
 & & q & \longmapsto & r
\end{array}
\tag{1.1.3}
$$

We claim that $[p]$ does not depend upon the choice the element $q \in Q^n$ mapping to r, i.e. the association $r \mapsto [p]$ just described gives a well defined map $Z^n(R^\bullet) \to \mathrm{H}^{n+1}(P^\bullet)$. To that end, suppose $q' \in Q^n$ is another element such that $\psi(q') = r$. Then $\psi(q' - q) = 0$, whence there is a unique element $p^* \in P^n$ such that $\varphi(p^*) = q' - q$. It is easy to check that $dq' = \varphi(p + dp^*)$. Since $[p + dp^*] = [p]$, our claim is proved. Thus we have a well defined map

$$\delta' \colon Z^n(R^\bullet) \longrightarrow \mathrm{H}^{n+1}(P^\bullet).$$

We will now show that $\delta'(B^n(R^\bullet)) = 0$. To see this, suppose $r \in B^n(R^\bullet)$, say $r = dr^*$. Let $q^* \in Q^{n-1}$ be an element such that $\psi(q^*) = r^*$. Such a q^* exists because ψ is surjective. Let $q = dq^*$. Then $\psi(q) = r$. Further, $dq = d^2q^* = 0$. The unique element $p \in P^{n+1}$ mapping to dq is therefore $p = 0$. Since $\delta'(r) = [p] = 0$, we are done. One consequence of this observation is that we now have a well defined map

$$\delta = \delta^n \colon \mathrm{H}^n(R^\bullet) \longrightarrow \mathrm{H}^{n+1}(P^\bullet) \tag{1.1.4}$$

such that the following diagram commutes

$$
\begin{array}{ccc}
Z^n(R^\bullet) & & \\
\downarrow & \searrow^{\delta'} & \\
\mathrm{H}^n(R^\bullet) & \xrightarrow[\delta]{} & \mathrm{H}^{n+1}(P^\bullet)
\end{array}
$$

The connecting maps turn out to be *functorial.* This means the following. Suppose we have a commutative diagram of complexes

$$
\begin{CD}
0 @>>> P'^\bullet @>{\varphi'}>> Q'^\bullet @>{\psi'}>> R'^\bullet @>>> 0 \\
@. @A{\alpha}AA @A{\beta}AA @A{\gamma}AA @. \\
0 @>>> P^\bullet @>>{\varphi}> Q^\bullet @>>{\psi}> R^\bullet @>>> 0
\end{CD}
\tag{1.1.5}
$$

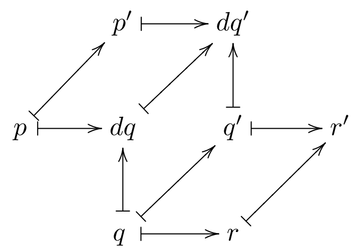
with the two rows being short exact sequences of complexes. Then the following diagram commutes for every $n \in \mathbf{Z}$.

$$
\begin{CD}
\mathrm{H}^n(R'^\bullet) @>{\delta}>> \mathrm{H}^{n+1}(P'^\bullet) \\
@A{\gamma_*}AA @AA{\alpha_*}A \\
\mathrm{H}^n(R^\bullet) @>>{\delta}> \mathrm{H}^{n+1}(P^\bullet)
\end{CD}
\tag{1.1.6}
$$

The proof of the commutativity of (1.1.6) is straightforward. Let $[r] \in \mathrm{H}^n(R^\bullet)$ and let $[p] = \delta([r]) \in \mathrm{H}^{n+1}(P^\bullet)$. Then we can choose representatives r and p for the respective cohomology classes so that there is an element $q \in Q^n$ such that $dq = \varphi(p)$. Let r', q', p' be the images of r, p, and q in R'^n, Q'^n, and P'^{n+1}

respectively. The proof is obtained by contemplating the following 3D version of diagram (1.1.3).

(1.1.7)



1.2. **The long exact sequence associated to a short exact sequence of complexes.** The connecting map δ of (1.1.4) is part of a long exact sequence of modules associated to the short exact sequence of complexes (1.1.1). Here is the precise statement.

Theorem 1.2.1. *Given the short exact sequence of complexes* (1.1.1) *the sequence*

$$\dots \xrightarrow{\psi_*} \mathrm{H}^{n-1}(R^\bullet) \xrightarrow{\delta} \mathrm{H}^n(P^\bullet) \xrightarrow{\varphi_*} \mathrm{H}^n(Q^\bullet) \xrightarrow{\psi_*} \mathrm{H}^n(R^\bullet) \xrightarrow{\delta} \mathrm{H}^{n+1}(P^\bullet) \xrightarrow{\varphi_*} \dots$$

is exact, where the maps labelled δ are the connecting homomorphisms (1.1.4). *This association of the long exact sequence above with* (1.1.1) *is functorial in the following sense: If we have a commutative diagram of complexes of the form* (1.1.5), *with the rows being exact sequence of complexes, then the following diagram commutes.*

$$\begin{array}{ccccccccccc}
\dots & \xrightarrow{\psi'_*} & \mathrm{H}^{n-1}(R'^\bullet) & \xrightarrow{\delta} & \mathrm{H}^n(P'^\bullet) & \xrightarrow{\varphi'_*} & \mathrm{H}^n(Q'^\bullet) & \xrightarrow{\psi'_*} & \mathrm{H}^n(R'^\bullet) & \xrightarrow{\delta} & \dots \\
 & & \uparrow{\scriptstyle \gamma_*} & & \uparrow{\scriptstyle \alpha_*} & & \uparrow{\scriptstyle \beta_*} & & \uparrow{\scriptstyle \gamma_*} & & \\
\dots & \xrightarrow[\psi_*]{} & \mathrm{H}^{n-1}(R^\bullet) & \xrightarrow[\delta]{} & \mathrm{H}^n(P^\bullet) & \xrightarrow[\varphi_*]{} & \mathrm{H}^n(Q^\bullet) & \xrightarrow[\psi_*]{} & \mathrm{H}^n(R^\bullet) & \xrightarrow[\delta]{} & \dots
\end{array}$$

Proof. The second part of the theorem is straightforward. The squares not involving δ commute since H^n respects compositions of maps of complexes and so must respect commutative diagrams of complexes (see equation (2.4.2) of Lecture 16). The squares that do involve the connecting maps δ commute by (1.1.6).

We only have to prove the first part. First let us show that the displayed sequence of maps forms a complex. Since $\psi \circ \varphi = 0$ therefore $\psi_* \circ \varphi_* = (\psi \circ \varphi)_* = 0$. It remains to prove that $\varphi_* \circ \delta = 0$. This is a simple consequence of the schematic diagram (1.1.3). In greater detail, suppose $x \in \mathrm{H}^n(R^\bullet)$, and say $x = [r]$ with $r \in Z^n(R^\bullet)$. Let r, q, p be as in (1.1.3), so that $\delta(x) = [p]$. Then $\varphi_*(\delta(x)) = \varphi_*[p] = [\varphi(p)] = [dq] = 0$.

Exactness at $\mathrm{H}^n(R^\bullet)$. Suppose $x \in \mathrm{H}^n(R^\bullet)$ is such that $\delta(x) = 0$. Write $x = [r]$ with $r \in Z^n(R^\bullet)$. Let q and p be as in diagram (1.1.3). Since $[p] = 0$, we see that there exists $p^* \in P^n$ such that $dp^* = p$. Let $q' = q - \varphi(p^*)$. Then q' is also a pre-image of r, and $dq' = dq - d\varphi(p^*) = dq - \varphi(dp^*) = dq - dq = 0$. Thus $q' \in Z^n(Q^\bullet)$, and $\psi_*([q']) = [r] = x$. This proves exactness at $\mathrm{H}^n(R^\bullet)$.

Exactness at $\mathrm{H}^n(Q^\bullet)$. Suppose $x \in \mathrm{H}^n(Q^\bullet)$ is such that $\psi_* x = 0$. Write $x = [q]$ for $q \in Z^n(Q^\bullet)$. Let$r = \psi(q)$. Since $\psi_* x = 0$, we have $r = \psi(q) \in B^n(R^\bullet)$, i.e. $r = dr^*$ for some $r^* \in R^{n-1}$. Pick a pre-image q^* of r^* in Q^{n-1}. Then $\psi(q - dr^*) = 0$, and hence there exists $p \in P^n$ such that $\varphi(p) = q - dr^*$. Since $d(q - r^*) = dq - d^2 r^* = 0$

(since $q \in Z^n(Q^\bullet)$), we see, by the injectivity of φ that $dp = 0$. Thus $p \in Z^n(P^\bullet)$ and clearly $\varphi_*[p] = [q]$. This proves exactness at $\mathrm{H}^n(Q^\bullet)$.

Exactness at $\mathrm{H}^{n+1}(P^\bullet)$. Let $x \in \mathrm{H}^{n+1}(P^\bullet)$ be such that $\varphi_* x = 0$. Pick $p \in Z^{n+1}(P^\bullet)$ such that $x = [p]$. Since $\varphi_* x = 0$, $\varphi(p) \in B^{n+1}(Q^\bullet)$. Thus $\varphi(p) = dq$ for some $q \in Q^n$. Let $r = \psi(q)$. Then $dr = d(\psi(q)) = \psi(dq) = \psi(\varphi(p)) = 0$. The elements r, q, dq, p fit into the schematic diagram (1.1.3), and hence $\delta[r] = [p] = x$, proving exactness at $\mathrm{H}^{n+1}(P^\bullet)$. □

Proposition 1.2.2. (The Snake Lemma) *Let*

(#)
$$
\begin{array}{ccccccccc}
 & & P & \xrightarrow{\varphi} & Q & \xrightarrow{\psi} & R & \longrightarrow & 0 \\
 & & {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\
0 & \longrightarrow & P' & \xrightarrow[\varphi']{} & Q' & \xrightarrow[\psi']{} & R' & &
\end{array}
$$

be a commutative diagram with exact rows. Then we have an A-map $\delta\colon \ker\gamma \to \operatorname{coker}\alpha$ *such that the sequence*

(##)
$$\ker\alpha \longrightarrow \ker\beta \longrightarrow \ker\gamma \xrightarrow{\delta} \operatorname{coker}\alpha \longrightarrow \operatorname{coker}\beta \longrightarrow \operatorname{coker}\gamma$$

is exact. Moreover, if φ is injective then the first map $\ker\alpha \to \ker\beta$ *is injective, and if ψ' is surjective, the last map* $\operatorname{coker}\beta \to \operatorname{coker}\gamma$ *is surjective.*

The exact sequence (##) is functorial in the diagram (#).

Proof. This is an easy consequence of Theorem 1.2.1. The details are left as an exercise. Some pointers: if φ is injective and ψ' is surjective, then the Proposition is in fact an immediate consequence of Theorem 1.2.1. Reduce to this case by replacing P with its image in Q, and R' with the image of Q' in R'. Justify the reduction. □

1.2.3. In reducing the Snake Lemma to a special case of Theorem 1.2.1, it might be useful to note that if we have a pair of A-maps $M \to N \to T$, with $M \to N$ surjective, then $\ker(M \to T)$ surjects on to $\ker(N \to T)$. In fact the latter can be identified with $\ker(M \to T)/\ker(M \to N)$. The dual statement is also useful for the recommended reduction, the dual statement being that if $T \to N \to M$ is a pair of maps such that $N \to M$ is injective, then $\operatorname{coker}(T \to N)$ injects into $\operatorname{coker}(T \to M)$.