LECTURE 16

Date of Lecture: March 10, 2022

Throughout this lecture A is a ring.

1. More primary decompositions

The main results we will be using are Theorems 1.2.8 (Existence), 1.2.9 (First Uniqueness Theorem), and 1.2.10 (Second Uniqueness Theorem) of Lecture 15.

1.1. An *irredundant primary decomposition* is the same as a reduced primary decomposition. Many authors prefer the term irredundant. I too prefer it (only because that is what I am used to), and so will likely use it more often than "reduced".

Lemma 1.1.1. *Assume A is Noetherian. Let $\mathfrak{m}$ be a maximal ideal of A, and $\mathfrak{a}$ an ideal such that $\sqrt{\mathfrak{a}} = \mathfrak{m}$. Then $\mathfrak{a}$ is $\mathfrak{m}$-primary. In particular, $\mathfrak{m}^n$ is $\mathfrak{m}$-primary for every $n \geq 1$.*

Proof. Note that $A/\mathfrak{a} \neq 0$, for $\mathfrak{a} \subset \mathfrak{m} \subsetneqq A$. Next, $\mathfrak{m} = \sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{q} \in V(\mathfrak{a})} \mathfrak{q}$, whence $\mathfrak{m}$ is minimal amongst members of $V(\mathfrak{a})$. On the other hand $\mathfrak{m}$ is a maximal ideal, and hence it is the only member of $V(\mathfrak{a})$. Since $\mathrm{ann}_A(A/\mathfrak{a}) = \mathfrak{a}$, it follows from Proposition 1.1.1 of Lecture 15 that

$$\bigcup_{\mathfrak{p} \in \mathrm{Ass}_A(A/\mathfrak{a})} V(\mathfrak{p}) = \mathrm{Supp}_A(A/\mathfrak{a}) = V(\mathfrak{a}) = \{\mathfrak{m}\}.$$

Thus $\mathrm{Ass}_A(A/\mathfrak{a}) = \{\mathfrak{m}\}$. □

Examples 1.1.2. Fix a field k in what follows.

1. Let

$$A = \frac{k[X, Y, Z]}{\langle XY - Z^2 \rangle}.$$

Let x, y, and z denote the images of X, Y, and Z in A respectively. Let $\mathfrak{p} = \langle x, z \rangle$. Then $\mathfrak{p}$ is a prime ideal of A. However $\mathfrak{p}^2$ is not $\mathfrak{p}$-primary. Indeed $xy = z^2 \in \mathfrak{p}^2$, $x \notin \mathfrak{p}^2$, but no power of y is in $\mathfrak{p}^2$.

2. Let $A = k[X, Y]$ and $\mathfrak{a} = \langle X, Y^2 \rangle$. Let $\mathfrak{m} = \langle X, Y \rangle$. Then $\mathfrak{m}$ is a maximal ideal and clearly $\sqrt{\mathfrak{a}} = \mathfrak{m}$. Hence $\mathfrak{a}$ is $\mathfrak{m}$-primary. However $\mathfrak{a}$ is not a power of $\mathfrak{m}$.

3. Let $A = k[X, Y]$ and $\mathfrak{a} = \langle X^2, XY \rangle$. Then for every $n \geq 1$ we have

$$\mathfrak{a} = \langle X \rangle \cap \langle X^2, Y \rangle = \langle X \rangle \cap \langle X^2, XY, Y^n \rangle.$$

The two intersections on the right are both irredundant primary decompositions of $\mathfrak{a}$. Thus irredundant primary decompositions are not unique.

2. **Chain and cochain complexes**

2.1. **Definitions.** A *chain complex* $(C_\bullet, d_\bullet)$ *of A-modules* is a collection of A-modules $\{C^n \mid n \in \mathbf{Z}\}$ together with a sequence of A-maps

$$\dots \xrightarrow{d_{n+1}} C_{n+1} \xrightarrow{d_n} C_n \xrightarrow{d_{n-1}} C_{n-1} \xrightarrow{d_{n-2}} \dots \tag{2.1.1}$$

such that $d_{n-1} \circ d_n = 0$, $n \in \mathbf{Z}$. Elements of C_n are called *n-chains* or just *chains.* The maps d_n, $n \in \mathbf{Z}$ are called *boundary maps* or *differentials.* The n^{th} map d_n is sometimes called the n^{th} boundary map or the n^{th} differential of $C_\bullet$. We set

$$B_n(C_\bullet) := \operatorname{im}(d_n) \quad \text{and} \quad Z_n(C_\bullet) := \ker(d_{n-1}), \qquad n \in \mathbf{Z}. \tag{2.1.2}$$

When the context is clear, we write B_n and Z_n for $B_n(C_\bullet)$ and $Z_n(C_\bullet)$ respectively. The elements of Z_n are called *n-cycles* or simply *cycles*, and those of B_n, *n-boundaries* or *boundaries.* Since $d_n \circ d_{n+1} = 0$, $B_n \subset Z_n$ for $n \in \mathbf{Z}$. The n^{th} *homology module* $\mathrm{H}_n(C_\bullet)$ of $C_\bullet$ is

$$\mathrm{H}_n(C_\bullet) := Z_n(C_\bullet)/B_n(C_\bullet), \qquad n \in \mathbf{Z}. \tag{2.1.3}$$

We sometimes write $d_n^{C_\bullet}$ or $d_n(C_\bullet)$ for the n^{th} boundary map of $C_\bullet$, if we wish to emphasise the role of the complex $C_\bullet$. It is also very common to drop the index n and write d for d_n. In that case, the relation $d_{n-1} \circ d_n = 0$ is written is a more compact form as $d^2 = 0$.

An element $z \in Z_n$ is said to be *homologous* to an element $z' \in C_n$ is $z' - z \in B_n$. Note that in this case z' is actually a cycle, i.e. it lies in Z_n.

Dual to the notion of a chain complex is that of cochain complex. The two notions are equivalent. A *cochain complex* of A modules is a collection of A-modules $\{C^n \mid n \in \mathbf{Z}\}$ together with maps $d^n \colon C^n \to C^{n+1}$, $n \in \mathbf{Z}$, such that if $C_n := C^{-n}$ and $d_n = d^{-(n+1)}$, then $(C_\bullet, d_\bullet)$ is a chain complex. The cochain complex $C^\bullet$ is schematically represented in the following way:

$$\dots \xrightarrow{d^{n-2}} C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \xrightarrow{d^{n+1}} \dots \tag{2.1.4}$$

Elements of C^n are called *n-cochains* or just *chains.* The map d^n is called the n^{th} *coboundary* or the n^{th}*-differential.* For $n \in \mathbf{Z}$, we set $B^n(C^\bullet) = B_{-n}(C_\bullet)$ and $Z^n(C^\bullet) = Z_{-n}(C_\bullet)$. We often write Z^n and B^n for $Z^n(C^\bullet)$ and $B^n(C^\bullet)$ respectively. The elements of C^n are called *n-cocycles*, or simply *cocycles*, and those of B^n, *n-coboundaries* or *boundaries.* The following relations are obvious:

$$B^n(C^\bullet) = \operatorname{im}(d^{n-1}) \quad \text{and} \quad Z^n(C^\bullet) = \ker(d^n), \qquad n \in \mathbf{Z}. \tag{2.1.5}$$

The n^{th} *cohomology module* $\mathrm{H}^n(C^\bullet)$ of $C^\bullet$ is

$$\mathrm{H}^n(C^\bullet) := Z^n(C^\bullet)/B^n(C^\bullet), \qquad n \in \mathbf{Z}. \tag{2.1.6}$$

Note that $\mathrm{H}^n(C^\bullet) = \mathrm{H}_{-n}(C_\bullet)$. As expected, an element $z \in Z^n$ is said to *cohomologous* to $z' \in C^n$ if $z - z' \in B^n$. Note that if this is the case, then z' is also an n-cocycle.

2.1.7. It is clear that given a chain complex there is an associated cochain complex, and vice-versa, and that there is really no essential difference between them. Such differences that exist are notational differences. The process of converting a chain complex into a cochain complex is that of "raising indices" ($(-)^n = (-)_{-n}$) and that of converting cochain complexes into chain complex is the process "lowering indices". (For differentials, one has to be a little more careful with indices.)

2.2. **Maps of complexes.** Let $C^\bullet$ and $D^\bullet$ be two cochain complexes. A colllection of maps $\varphi = \{\varphi^n \mid n \in \mathbf{Z}\}$ is said to be a *cochain map* between $C^\bullet$ and $D^\bullet$, written $\varphi\colon C^\bullet \to D^\bullet$, if the following diagram commutes for every $n \in \mathbf{Z}$,

$$\begin{array}{ccc} C^n & \xrightarrow{d^n_{C^\bullet}} & C^{n+1} \\ {\scriptstyle \varphi^n}\big\downarrow & & \big\downarrow{\scriptstyle \varphi n+1} \\ D^n & \xrightarrow[d^n_{D^\bullet}]{} & D^{n+1} \end{array} \tag{2.2.1}$$

A *map of chain complexes* is defined in a similar manner. I leave it to you to define a map $\varphi\colon C_\bullet \to D_\bullet$ between to chain complexes $C_\bullet$ and $D_\bullet$.

It is clear that if $\varphi\colon C^\bullet \to D^\bullet$ and $\psi\colon D^\bullet \to E^\bullet$ are two cochain maps, then $\psi\circ\varphi := \{\psi^n \circ \varphi^n\}$ is a cochain map from $C^\bullet$ to $E^\bullet$. Similarly maps of chain complexes can also be composed to produce a chain complex. the upshot is that cochain complexes (respectively, chain complexes) of A-modules "form a category".

2.3. **Our conventions.** For us, a complex will mean a cochain complex. We will often refer to a cochain map as a "map of complexes". In view of the observations made in 2.1.7, by the simple trick of lowering indices, every result we prove for cochain complexes, proves one for chain complexes. The translations from one language to another is totally transparent.

2.4. **Cohomology as a functor.** Let $\varphi\colon C^\bullet \to D^\bullet$ be a map of complexes. Let $z \in Z^n(C^\bullet)$. Then $d^n(\varphi^n(z)) = \varphi^{n+1}(d^n(z)) = 0$, whence $\varphi^n(z) \in Z^n(D^\bullet)$. Similarly, if $b \in B^n(C^\bullet)$, so that $b = d^{n-1}x$ for some $x \in C^{n-1}$, then $\varphi^n(b) = d^{n-1}(\varphi^{n-1}(x))$, whence $\varphi^n(b) \in B^n(D^\bullet)$. We therefore have a well defined map of A-modules

$$\mathrm{H}^n(\varphi)\colon \mathrm{H}^n(C^\bullet) \longrightarrow \mathrm{H}^n(D^\bullet), \qquad [z] \longmapsto [\varphi^n(z)], \tag{2.4.1}$$

where $[z]$ (respectively $[\varphi^n(z)]$) is the coset $z + B^n(C^\bullet)$ (respectively $\varphi^n(z) + B^n(D^\bullet)$). It is easy to verify that if $\psi\colon D^\bullet \to E^\bullet$ is a second map of complexes then

$$\mathrm{H}^n(\psi) \circ \mathrm{H}^n(\varphi) = \mathrm{H}^n(\psi\circ\varphi), \qquad n \in \mathbf{Z}. \tag{2.4.2}$$

Equation (2.4.2) is way of saying that each H^n is a "functor" from the "category" of complexes of A-modules to the "category" Mod_A.

References

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[Ku] E. Kunz *Introduction to Commutative Algebra and Algebraic Geometry*, Birkhaüser, Boston, Basel, Berlin, 1985.