LECTURE 15

Date of Lecture: March 8, 2022

1. **Primary decomposition**

Throughout A is a ring and M an A-module.

1.1. **The support of M and $\mathrm{Ass}(M)$.** The basic proposition is the following:

Proposition 1.1.1. *Let A be Noetherian, $M \neq 0$ and finitely generated. Then*

$$\mathrm{Supp}(M) = \bigcup_{\mathfrak{p}\in \mathrm{Ass}(M)} V(\mathfrak{p}).$$

Proof. We will use the equality $\mathrm{Supp}(M) = V(\mathrm{ann}(M))$ established in Problem 2 of Homework 4.

Since $\mathrm{ann}(M) = \cap_{m\in M}\mathrm{ann}(m)$, we have $\mathrm{ann}(M) \subset \cap_{\mathfrak{p}\in\mathrm{Ass}(M)}\mathfrak{p}$. Since $V(-)$ reverses inclusions, it follows that $\mathrm{Supp}(M) = V(\mathrm{ann}(M)) \supset V(\cap_{\mathfrak{p}\in\mathrm{Ass}(M)}\mathfrak{p}) = \cup_{\mathfrak{p}\in\mathrm{Ass}(M)}V(\mathfrak{p})$.

Conversely, suppose $\mathfrak{q} \in \mathrm{Supp}(M)$. Then $M_{\mathfrak{q}} \neq 0$. By Problem 6 of Homework 4, $\mathrm{Ass}_{A_{\mathfrak{q}}}(M_{\mathfrak{q}})$ is non-empty. By Lemma 2.1.2 of Lecture 14, there exists $\mathfrak{p}_0 \in \mathrm{Ass}_A(M)$ such that $(\mathfrak{p}_0)_{\mathfrak{q}} \in \mathrm{Ass}_{A_{\mathfrak{q}}}(M_{\mathfrak{q}})$, and $\mathfrak{p}_0 \subset \mathfrak{q}$. Thus $\mathfrak{q} \in V(\mathfrak{p}_0)$, whence $\mathfrak{q} \in \cup_{\mathfrak{p}\in\mathrm{Ass}(M)}V(\mathfrak{p})$. This proves that $\mathrm{Supp}(M) \subset \cup_{\mathfrak{p}\in\mathrm{Ass}(M)}V(\mathfrak{p})$. □

Corollary 1.1.2. *With the above hypotheses*

$$\sqrt{\mathrm{ann}(M)} = \bigcap_{\mathfrak{p}\in\mathrm{Ass}(M)} \mathfrak{p}.$$

Proof. From the theorem and the fact that $\mathrm{Supp}(M) = V(\mathrm{ann}(M))$, we get

$$V(\mathrm{ann}(M)) = \cup_{\mathfrak{p}\in\mathrm{Ass}(M)}V(\mathfrak{p}) = V(\cap_{\mathfrak{p}\in\mathrm{Ass}(M)}\mathfrak{p}).$$

Now $\cap_{\mathfrak{p}\in\mathrm{Ass}(M)}\mathfrak{p}$ is a radical ideal, being the intersection of radical ideals. We use the fact that $V(\mathfrak{a}) = V(\mathfrak{b})$ for ideals $\mathfrak{a}$ and $\mathfrak{b}$ if and only if $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$ to arrive at the required conclusion.

We point out that the conclusion is true even if $M = 0$, for then both sides equal A, the right side because $\mathrm{Ass}(M) = \emptyset$. □

Definition 1.1.3. An element $a \in A$ is said to *nilpotent for M* is the A-endomorphism μ_a^n on M is zero for some $n \geq 1$. Here, as before, $\mu_a \colon M \to M$ is the A-endomorphism $m \mapsto am$, $m \in M$.

Clearly a is nilpotent for M if and only if $a \in \sqrt{\mathrm{ann}(M)}$. An immediate consequence is the following result.

Lemma 1.1.4. *Suppose A is Noetherian and M is finitely generated. The following are equivalent*

(a) *$a \in A$ is nilpotent for M;*
(b) *$a \in \bigcap_{\mathfrak{p}\in\mathrm{Ass}(M)} \mathfrak{p}$.*

Proof. This follows immediately from Corollary 1.1.2 and the fact that a is nilpotent for M if and only if $a \in \sqrt{\operatorname{ann}(M)}$. $\square$

1.2. **Primary submodules and primary decomposition.** We begin with a definition.

Definition 1.2.1. A submodule N of M is called *primary* if $\operatorname{Ass}(M/N)$ consists of a single element. In this case, if $\operatorname{Ass}(M/N) = \{\mathfrak{p}\}$, then N is said to be $\mathfrak{p}$*-primary*, or, if one is feeling expansive, *a* $\mathfrak{p}$*-primary submodule of* M.

Lemma 1.2.2. *If A is Noetherian and M finitely generated, then a submodule N of M is primary if and only if every zero divisor of M/N is nilpotent for M/N.*

Proof. Let $\mathrm{ZD}(M/N)$ be the set of zero divisors of M/N and $\mathrm{Nil}(M/N)$ the set of nilpotent elements in A for M/N. Clearly $\mathrm{Nil}(M/N) \subset \mathrm{ZD}(M/N)$. Therefore we have to show that N is primary if and only if $\mathrm{ZD}(M/N) = \mathrm{Nil}(M/N)$. We first note that by Problem 7 of Homework 4, $\mathrm{ZD}(M/N) \cup_{\mathfrak{p}\in\operatorname{Ass}(M/N)} \mathfrak{p}$. From Lemma 1.1.4, wehave $\mathrm{Nil}(M/N) = \cap_{\mathfrak{p}\in\operatorname{Ass}(M/N)}\mathfrak{p}$. Thus $\mathrm{ZD}(M/N) = \mathrm{Nil}(M/N)$ if and only if

$$\bigcup_{\mathfrak{p}\in\operatorname{Ass}(M/N)} \mathfrak{p} = \bigcap_{\mathfrak{p}\in\operatorname{Ass}(M/N)} \mathfrak{p}.$$

The above is true when and only when $\operatorname{Ass}(M/N) = \{\mathfrak{p}\}$. $\square$

Remark 1.2.3. There is a classical notion of primary ideals which predates the modern approach via associated primes. Classically, an ideal $\mathfrak{a}$ of A is said to be primary if it has the following property: If $a, b \in A$ are such that $ab \in \mathfrak{a}$, $b \notin \mathfrak{a}$, then $a^n \in \mathfrak{a}$ for some $n \geq 1$. Lemma 1.2.2 shows that when A is Noetherian, $\mathfrak{a}$ is primary in the classical sense if and only if $\mathfrak{a}$ is a primary submodule of A. The modern treatment through associated primes is due to Bourbaki.

Recall from Problem 3 of Homework 4 that a submodule N of M is said to be irreducible if two submodules N_1 and N_2 of A have intersection equal to N only when one of them equals N.

Proposition 1.2.4. *Let A be Noetherian, M finitely generated, and N a proper submodule of M (i.e. $N \neq M$). If N is irreducible then N is primary.*

Proof. By Problem 6 of Homework 4, $\operatorname{Ass}(M/N) \neq \emptyset$, for $M/N \neq 0$. Suppose N is irreducible. Let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be two elements of $\operatorname{Ass}(M/N)$. By problem 4 (c) of Homework 4, there exist submodules N_1/N and N_2/N of M/N such that $N_i/N \cong A/\mathfrak{p}_i$, $i = 1, 2$. Since $A/\mathfrak{p}_i \neq 0$, $N \subsetneqq N_i$, for $i = 1, 2$. The irreducibility of N then ensures that $N \subsetneqq N_1 \cap N_2$. Let $x \in (N_1 \cap N_2) \smallsetminus N$ and let $\bar{x}$ be its image in M/N. Now $0 \neq \bar{x} \in (N_1/N) \cap (N_2/N)$. Since $N_i/N \cong A/\mathfrak{p}_i$, $\operatorname{ann}(\bar{x}) = \mathfrak{p}_i$, $i = 1, 2$. It follows that $\mathfrak{p}_1 = \mathfrak{p}_2$, i.e. $\operatorname{Ass}(M/N)$ has exactly one element. $\square$

For Noetherian rings we have the following sastisfying result.

Lemma 1.2.5. *If A is Noetherian then the intersection of finitely many $\mathfrak{p}$-primary submodules of M is $\mathfrak{p}$-primary.*

Proof. It is enough to prove that the intersection of two $\mathfrak{p}$-primary submodules of M is again $\mathfrak{p}$-primary. So suppose N_1 and N_2 are $\mathfrak{p}$-primary submodules of M. If $N_1 \subset N_2$, there is nothing to prove. So assume that $N_1 \cap N_2 \not\subseteq N_1$, i.e. $N_1/(N_1 \cap N_2) \neq 0$. Using the fact that $N_1/(N_1 \cap N_2) \cong (N_1 + N_2)/N_2 \subset M/N_2$, we see that $\operatorname{Ass}(N_1/(N_1 \cap N_2)) \subset \operatorname{Ass}(M/N_2) = \{\mathfrak{p}\}$. (We are using 5 (a) of Homework

4 to arrive at this conclusion.) Since A is Noetherian and $N_1/(N_1 \cap N_2) \neq 0$, by Problem 6 of Homework 4, $\mathrm{Ass}(N_1/(N_1 \cap N_2)) \neq \emptyset$, whence $\mathrm{Ass}(N_1/(N_1 \cap N_2)) = \{\mathfrak{p}\}$. Consider the exact sequence

$$0 \longrightarrow N_1/(N_1 \cap N_2) \longrightarrow M/(N_1 \cap N_2) \longrightarrow M/N_1 \longrightarrow 0.$$

By 5 (b) of Homework 4, we see that $\mathrm{Ass}(M/(N_1 \cap N_2)) \subset \{\mathfrak{p}\}$, i.e. either $\mathrm{Ass}(M/(N_1 \cap N_2)) = \emptyset$ or $\mathrm{Ass}(M/(N_1 \cap N_2)) = \{\mathfrak{p}\}$. Using the fact that A is Noetherian and the fact that $M/(N_1 \cap N_2) \neq 0$ the first possibility can be eliminated (see Problem 6 of Homework 4). □

Here is the important definition (see [Ku, pp.179–180, Definition 2.16]):

Definition 1.2.6. A submodule N of M is said to have a *primary decomposition* if there exist primary submodules $N_1, \dots, N_d$ of M such that

$$(*) \qquad N = N_1 \cap \cdots \cap N_d.$$

The primary decomposition $(*)$ is said to be *reduced* if

- **(a)** If N_i is $\mathfrak{p}_i$-primary $(i = 1, \dots, d)$, then the $\mathfrak{p}_i$ are distinct for $i = 1, \dots, d$.
- **(b)** $\bigcap_{i \neq j} \mathfrak{p}_i \not\subseteq \mathfrak{p}_i$ for $i = 1, \dots d$.

1.2.7. In view of Lemma 1.2.5 it is clear that if A is Noetherian and N is a submodule of M which has a primary decomposition, then N has a reduced primary decomposition. Indeed the Lemma allows to write a primary decomposition $(*)$ of N which satisfies **(a)** above. The condition **(b)** is easily met by dropping any N_i which lies in the intersection of the remaining primary submodules ocurring in $(*)$.

Theorem 1.2.8. (Existence) *Let A be Noetherian and M finitely generated. Every submodule module N of M such that $N \neq M$ has a reduced primary decomposition.*

Proof. This is Problem 3 of Homework 4 combined with Proposition 1.2.4 and the observation in 1.2.7. □

Theorem 1.2.9. (The First Uniqueness Theorem)*Let A be Noetherian, M finitely generated, and N a submodule of M. Let $N = N_1 \cap \cdots \cap N_d$ be a reduced primary decomposition of N with $\mathrm{Ass}(M/N_i) = \{\mathfrak{p}_i\}$, $i = 1, \dots, d$. Then $\mathrm{Ass}(M/N) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_d\}$.*

Proof. Let $H_i = \cap_{j \neq i} \mathfrak{p}_j$, for $i = 1, \dots, d$. Since $0 \neq H_i/N \cong (H_i + N_i)/N_i \subset M/N_i$, we have $\emptyset \neq \mathrm{Ass}(H_i/N) \subset \mathrm{Ass}(M/N_i) = \{\mathfrak{p}_i\}$. Thus $\mathrm{Ass}(H_i/N) = \{\mathfrak{p}_i\}$. Since $\mathrm{Ass}(H_i/N) \subset \mathrm{Ass}(M/N)$, $\mathfrak{p}_i \in \mathrm{Ass}(M/N)$ for $i = 1, \dots d$. It follows that $\{\mathfrak{p}_1, \dots, \mathfrak{p}_d\} \subset \mathrm{Ass}(M/N)$.

To show that $\mathrm{Ass}(M/N) \subset \{\mathfrak{p}_1, \dots, \mathfrak{p}_d\}$, we proceed by induction. If $d = 1$, the statement is obvious. Suppose it is true for any reduced primary decomposition with $d - 1$ primary submodules in the decomposition. Then $\mathrm{Ass} M/H_i \subset \{\{\mathfrak{p}_j \mid J \neq i\}$. From the exact sequence

$$0 \longrightarrow H_i/N \longrightarrow M/N \longrightarrow M/H_i \longrightarrow 0$$

we get $\mathrm{Ass}(M/N) \subset \mathrm{Ass}(H_i/N) \cup \mathrm{Ass}(M/H_i) \subset \{\mathfrak{p}_1, \dots, \mathfrak{p}_d\}$. We are once again using the isomorphism $H_i/N \cong (H_i + N_i)/N_i$ for $i = 1, \dots, d$ and our earlier computation that $\mathrm{Ass}((H_i + N_i)/N_i) = \{\mathfrak{p}_i\}$. □

Before stating the second uniqueness theorem, i.e.Theorem 1.2.11 below, I would like to discuss the behaviour of reduced primary decompositions under localization. To that end, suppose N is a $\mathfrak{p}$-primary submodule of M and S a multiplicative system in A. According to Lemma 2.1.2 of Lecture 14, $\mathrm{Ass}(S^{-1}(M/N)) = \{S^{-1}\mathfrak{p}\}$, whence $S^{-1}N$ is an $S^{-1}\mathfrak{p}$-primary submodule of $S^{-1}M$. Moreover, we claim that in this case, if $m \in M$ is such that $m/1 \in S^{-1}N$, then $m \in N$. To see this, suppose $m/1 = x/s$ where $x \in N$ and $s \in S$. Then, there exists $t \in S$ such that $tsm = tx$. Suppose $m \notin N$. Then ts is a zero divisor for M/N (see Problem 7 of Homework 4), which in turn means that $ts \in \mathfrak{p}$ contradicting the fact that $\mathfrak{p}_i \cap S = \emptyset$. Thus $m \in N$.

Now suppose $\mathfrak{p} \cap S \neq \emptyset$. Assume further that A is Noetherian and M is finitely generated. Since $\mathfrak{p} \cap S \neq \emptyset$, S has a zero divisor of M/N, say s. By Lemma 1.2.2, s is nilpotent for M/N. This means the unit $s/1$ in $S^{-1}A$ is nilpotent for $S^{-1}(M/N)$. It is immediate that $S^{-1}(M/N) = 0$, i.e. $S^{-1}N = S^{-1}M$.

We are ready to make a formal statement of what we just discussed.

Proposition 1.2.10. *Let A be Noetherian, M finitely generated, and N a submodule of M. Let S be a multiplicative system in A.*

(a) *Suppose N is $\mathfrak{p}$-primary (as a submodule of M). If $\mathfrak{p} \cap S = \emptyset$, then $S^{-1}N$ is an $S^{-1}\mathfrak{p}$-primary submodule of $S^{-1}M$. In this case, if $m \in M$ is such that $m/1 \in S^{-1}N$, then $m \in N$. If $\mathfrak{p} \cap S \neq \emptyset$, then $S^{-1}N = S^{-1}M$.*

(b) *If $N = \bigcap_{i=1}^{d} N_i$ is a reduced primary decomposition of N, with N_i being $\mathfrak{p}_i$-primary submodules of M, then*

$$S^{-1}N = \bigcap_{\mathfrak{p}_i \cap S = \emptyset} S^{-1}N_i$$

is a reduced primary decomposition of $S^{-1}N$ as a submodule of $S^{-1}M$.

Proof. We only have to prove (b), since (a) has been proved in the discussion leading to this proposition. Part (b) is an immediate consequence of the fact hat localization commutes with intersection. □

Theorem 1.2.11. (The Second Uniqueness Theorem) *Let A be Noetherian, M finitely generated, N a submodule of M. Suppose*

$$(\dagger) \qquad N = \bigcap_{i=1}^{d} N_i$$

is a reduced primary decomposition of N. For $\mathfrak{p} \in \mathrm{Ass}(M/N)$, let $N(\mathfrak{p})$ be the unique $\mathfrak{p}$-primary submodule occurring in the collection $\{N_1, \ldots, N_d\}$. If $\mathfrak{p}$ is a minimal prime ideal in $\mathrm{Ass}(M/N)$ then $N(\mathfrak{p})$ is the inverse image of $N_{\mathfrak{p}}$ under the localization map $M \to M_{\mathfrak{p}}$.

Orienting remark. One important consequence of the theorem is that if $\mathfrak{p}$ is a minimal associated prime of M/N, then the $\mathfrak{p}$-primary component of N in any reduced primary decomposition of N is uniquely determined. This property need not hold if $\mathfrak{p}$ is not a minimal element in $\mathrm{Ass}(M/N)$. If all the primes in $\mathrm{Ass}(M/N)$ are minimal, then N has a unique reduced primary decomposition.

Proof. Let $S = A \smallsetminus \mathfrak{p}$. If $\mathfrak{q} \in \mathrm{Ass}(M/N)$ is such that $\mathfrak{q} \cap S = \emptyset$ then $\mathfrak{q} \subset \mathfrak{p}$, whence $\mathfrak{q} = \mathfrak{p}$ by the minimality of $\mathfrak{p}$. Part (b) of Proposition 1.2.10 then implies that $N_{\mathfrak{p}} = N(\mathfrak{p})_{\mathfrak{p}}$. The required result follows from part (a) of the proposition. □

References

[AM] M.F. Atiyah and I.G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, London, 1969.

[Ku] E. Kunz *Introduction to Commutative Algebra and Algebraic Geometry*, Birkhaüser, Boston, Basel, Berlin, 1985.