LECTURE 14

Date of Lecture: March 3, 2022

1. The Cohen-Seidenberg Going-Up Theorem

1.1. **The Going-up Theorem.** Throughout this subsection, we will assume $A \subset B$ is an integral extension. Let us recall the six results we proved for the integral this integral extension in § 3.2 of Lecture 13. We keep the numbering as in *loc.cit.*

1. Let A and B be integral domains. Then A is a field if and only if B is a field.
2. Let S be a multiplicative system in A. Then $S^{-1}A \subset S^{-1}B$ is also an integral extension. (We are using the fact that localization is exact to identify $S^{-1}A$ with a subring of $S^{-1}B$.)
3. Let $\mathfrak{b}$ be an ideal in B and $\mathfrak{a} = \mathfrak{b}^c$, so that the natural map $A/\mathfrak{a} \to B/\mathfrak{b}$ is injective. Regard $A/\mathfrak{a}$ as a subring of $B/\mathfrak{b}$ is this manner. Then $B/\mathfrak{b}$ is integral over A.
4. Suppose A is a local ring with maximal ideal $\mathfrak{m}$. A prime ideal $\mathfrak{q}$ in B is maximal if and only if $\mathfrak{q}^c = \mathfrak{m}$.
5. If A is a field then all prime ideals of B are maximal.
6. The map $\phi\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ induced by $A \subset B$ is surjective.

To the above list, we add three more results.

7. The map $\phi\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ of **6** is a closed map.
Proof. Let Z be a closed subset of $\mathrm{Spec}(B)$, say $Z = V(\mathfrak{b})$, where $\mathfrak{b}$ is an ideal of B. Let $\mathfrak{a} = \mathfrak{b}^c$. By **3**, $A/\mathfrak{a} \subset B/\mathfrak{b}$ is an integral extension and hence the induced map $\mathrm{Spec}(B/\mathfrak{b}) \to \mathrm{Spec}(A/\mathfrak{a})$ is surjective by **6**. Now $Z = V(\mathfrak{b})$ can be identified with $Spec(B/\mathfrak{b})$ and $V(\mathfrak{a})$ with $\mathrm{Spec}(A/\mathfrak{a})$ and we have just proved that $\phi(Z) = V(\mathfrak{a})$. Thus ϕ is a closed map. □

8. If $\mathfrak{q}_1 \subset \mathfrak{q}_2$ is a chain of prime ideals in B such that $\mathfrak{q}_1^c = \mathfrak{q}_1^c$ then $\mathfrak{q}_1 = \mathfrak{q}_2$.
Proof. Let $\mathfrak{p}$ be the common contraction of $\mathfrak{q}_1$ and $\mathfrak{q}_2$ to A. By a now familiar trick, using (2.2.1) and (2.2.2) of Lecture 13, we can assume A is a local ring and $\mathfrak{p}$ its maximal ideal, by localizing A at $\mathfrak{p}$ if necessary. By **4** we see that both $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are maximal ideals of B. The assertion follows. □

9. (Going-Up) Let $\mathfrak{p}_0 \subset \mathfrak{p}_1$ be a chain of prime ideals in A, and $\mathfrak{q}_0$ a prime ideal of B lying over $\mathfrak{p}_0$. Then there exists $\mathfrak{q}_1 \in \mathrm{Spec}(B)$ such that $\mathfrak{q}_0 \subset \mathfrak{q}_1$ and $\mathfrak{q}_1$ lies over $\mathfrak{p}_1$.
Proof. We may assume, by localizing A at $\mathfrak{p}_1$ if necessary, and using (2.2.1) and (2.2.2) of Lecture 13, that A is a local ring with maximal ideal $\mathfrak{p}_1$. Let $\mathfrak{q}_1$ be a maximal ideal of B containing $\mathfrak{q}_0$ (there always exists one by Zorn's Lemma, along the lines the first lemma of Lecture 2). According to **4**, $\mathfrak{q}_1$ must contract to $\mathfrak{p}_1$. □

We are now in a position to state a part of the results of Cohen and Seidenberg [CS] (the paper can be accessed here).

Theorem 1.1.1. *Let $A \subset B$ be an integral extension of rings and $\phi\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ the induced map.*

(a) *The map ϕ is surjective.*
(b) *The map ϕ is closed.*
(c) *If $\mathfrak{q}_1 \subset \mathfrak{q}_2$ is a chain of prime ideals in B such that $\mathfrak{q}_1^c = \mathfrak{q}_1^c$ then $\mathfrak{q}_1 = \mathfrak{q}_2$.*
(d) (The Going-Up Theorem of Cohen-Seidenberg) *Let*

$$(\#) \qquad \mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \cdots \subset \mathfrak{p}_n$$

be a chain of prime ideals in A, and

$$(\dagger) \qquad \mathfrak{q}_1 \subset \mathfrak{q}_2 \subset \cdots \subset \mathfrak{q}_m$$

a chain of prime ideals in B, with $1 \le m \le n$, such that $\mathfrak{q}_i$ lies over $\mathfrak{p}_i$ for $i = 1, \ldots, m$. Then the chain $(\dagger)$ can be extended to a chain of prime ideals $\mathfrak{q}_1 \subset \mathfrak{q}_2 \subset \cdots \subset \mathfrak{q}_n$ with $\mathfrak{q}_i$ lying over $\mathfrak{p}_i$ for $i = 1, \ldots, n$.

2. Associated primes

2.1. Let A be a ring and $M \in \mathrm{Mod}_A$. Recall from Homework 4 that $\mathfrak{p} \in \mathrm{Spec}(A)$ is an associated prime of M is there exists $m \in M$, $m \neq 0$, such that $\mathfrak{p} = \mathrm{ann}(m)$.

One should point out that the concept of associated primes only makes sense for non-zero rings and modules.

Lemma 2.1.1. *Let $M \neq 0$ be a finitely generated module over a Noetherian ring A. If $\mathfrak{a}$ is an ideal of A consisting only of zero divisors of M, then there is an $m \in M$, $m \neq 0$, such that $\mathfrak{a}m = 0$.*

Proof. By Problem 7 of Homework 4, $\mathfrak{a} \subset \cup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}$. By 8 (b) of *loc.cit.*, $\mathrm{Ass}(M)$ is a finite set, whence $\mathfrak{a} \subset \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Ass}(M)$ (see the first proposition on p.1 of Lecture 8). The result follows. ☐

Lemma 2.1.2. *Let A be a Noetherian ring, S a multiplicative system in A, and M an A-module. Then*

$$\mathrm{Ass}(S^{-1}M) = \{S^{-1}\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Ass}(M),\ \mathfrak{p} \cap S = \emptyset\}.$$

Proof. Suppose $\mathfrak{p} \in \mathrm{Ass}(M)$ and $\mathfrak{p} \cap S = \emptyset$. Then there exists $m \in M \smallsetminus \{0\}$ such that $\mathfrak{p} = \mathrm{ann}(m)$. It is immediate that $S^{-1}\mathfrak{p} \subset \mathrm{ann}(m/1)$. Now suppose $a/s \in \mathrm{ann}(m/1)$. Since $a/s = (s/1)^{-1}(a/1)$ and $s/1$ is a unit in $S^{-1}A$, therefore $(a/1)(m/1) = 0$. This means there exists $t \in S$ such that $tam = 0$, i.e. $ta \in \mathrm{ann}(m) = \mathfrak{p}$. Now $t \notin \mathfrak{p}$, and so $a \in \mathfrak{p}$ and hence $a/s \in S^{-1}\mathfrak{p}$. Thus $S^{-1}\mathfrak{p} = \mathrm{ann}(m/1)$. In other words, if $\mathfrak{p} \in \mathrm{Ass}(M)$ and $S \cap \mathfrak{p} = \emptyset$, then $S^{-1}\mathfrak{p} \in \mathrm{Ass}(S^{-1}M)$.

Conversely, suppose $Q \in \mathrm{Ass}(S^{-1}M)$, say $Q = \mathrm{ann}(m/t)$ for some $m \in M$ and $t \in S$. Since $t/1$ is a unit in A, $\mathrm{ann}(m/t) = \mathrm{ann}(m/1)$, and hence $Q = \mathrm{ann}(m/1)$. According to Problem 6 of HW 1, $Q = S^{-1}\mathfrak{p}$ for a unique prime $\mathfrak{p}$ in A and in this case, $\mathfrak{p} \cap S = \emptyset$. In fact, by *loc.cit.*, $\mathfrak{p} = \{a \in A \mid a/1 \in Q\}$. We have to show that $\mathfrak{p} \in \mathrm{Ass}(M)$. Since A is Noetherian, we have a finite number of elements $p_1, \ldots, p_d \in \mathfrak{p}$ which generate $\mathfrak{p}$ as an ideal. Now $p_i/1 \in S^{-1}\mathfrak{p} = \mathrm{ann}(m/1)$, for $i = 1, \ldots, d$. So there exist $s_i \in S$ such that $s_i p_i m = 0$ for $i = 1, \ldots, d$. Let $s = s_1 \ldots s_d$. Every p_i lies in $\mathrm{ann}(sm)$, and so $\mathfrak{p} \subset \mathrm{ann}(sm)$. Moroever, if $a \in \mathrm{ann}(sm)$, then $a/1 \in \mathrm{ann}(sm/1) = \mathrm{ann}(m/1) = S^{-1}\mathfrak{p}$. This means $a \in \mathfrak{p}$. Thus $\mathfrak{p} = \mathrm{ann}(sm)$, and so $\mathfrak{p} \in \mathrm{Ass}(M)$. ☐

References

[AM] M.F. Atiyah and I.G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, London, 1969.

[CS] I.S. Cohen and A. Seidenberg, Prime ideals and integral dependence, *Bull. Amer. Math. Soc.*, **52**, 1946, 252–261.