LECTURE 13

Date of Lecture: March 1, 2022

For a module M over a ring A, and an ideal $\mathfrak{a}$ of A, we have the well-known relation

$$M/\mathfrak{a}M = M \otimes_A A/\mathfrak{a}. \tag{\#}$$

The universal A-bilinear map $B_u\colon M \times A/\mathfrak{a} \to M/\mathfrak{a}M$ is $(m, a+\mathfrak{a}) \mapsto am + \mathfrak{a}M$. B_u is well defined, as is readily verified (a proof at the speed of light is: if $a + \mathfrak{a} = 0 + \mathfrak{a}$, i.e. if $a \in \mathfrak{a}$, then $am + \mathfrak{a}M = 0$). Any A-bilinear map $B\colon M \times A/\mathfrak{a} \to T$ must necessarily send all elements of the form $(am, a' + \mathfrak{a})$, $a \in \mathfrak{a}$, to zero, whence the map $\varphi_B\colon M/\mathfrak{a}M \to T$, $m + \mathfrak{a}M \mapsto B(m, 1 + \mathfrak{a})$ is well defined. It is clear that φ_B is the unique map such that $\varphi_B \circ B_u = B$. Once again, I would like to stress the importance of universal properties, for this allows us to write (#) as an equality rather than as an isomorphism.

1. Extension and contraction of ideals

1.1. **The ideals $\mathfrak{a}^e$ and $\mathfrak{b}^c$.** We follow the notations of Atiyah-Macdonald [AM] in this subsection (see [AM, pp.9–10]). Let $f\colon A \to B$ be a ring homomorphism. For an ideal $\mathfrak{b}$ in B, we often write $\mathfrak{b}^c$ for the ideal $f^{-1}(\mathfrak{b})$ in A. The ideal $\mathfrak{b}^c$ is called the *contraction of* $\mathfrak{b}$ to A, explaining the superscript "c". Similarly, if $\mathfrak{a}$ is an ideal in A, we write $\mathfrak{a}^e$ for the ideal $\mathfrak{a}B$ of B generated by images of elements in $\mathfrak{a}$, and $\mathfrak{a}^e$ is called the *extension of* $\mathfrak{a}$ to B. I prefer the symbol $\mathfrak{a}B$ over $\mathfrak{a}^e$, but I might use the latter notation occasionally. Note that $\mathfrak{a}B$ consists of finite sums of the form $\sum_i a_i b_i$ with $a_i \in \mathfrak{a}$ and $b_i \in B$, the product $a_i b_i$ being scalar multiplication in the A-module B.

Note that $\mathfrak{a}B$ is the smallest ideal in B containing $f(\mathfrak{a})$.

If A is a subring of B, f the inclusion map $A \subset B$, and $\mathfrak{b}$ an ideal in B , then clearly $\mathfrak{b}^c = \mathfrak{b} \cap A$, explaining the term "contraction". In this case, $\mathfrak{a}$ is an ideal in A, $\mathfrak{a}^e$ is the smallest ideal in B containing $\mathfrak{a}$.

It is clear that $\mathfrak{q}^c$ is a prime ideal in A if $\mathfrak{q}$ is a prime ideal in B. In fact we have proven this in class. The map

$$^a f\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

induced by f, and defined on page 4 of Lecture 8, is precisely the map $\mathfrak{q} \mapsto \mathfrak{q}^c$.

Definition 1.1.1. A prime ideal $\mathfrak{q}$ of B is said to *lie over* a prime ideal $\mathfrak{p}$ of A if $\mathfrak{q}^c = \mathfrak{p}$. In this case we say $\mathfrak{p}$ *lies below* or *lies under* $\mathfrak{q}$.

1.2. **The inverse image of a closed set.** In the above situation, if $\mathfrak{a}$ is an ideal in A, and $\mathfrak{q}$ is a prime ideal in B such that $\mathfrak{q}^c$ contains $\mathfrak{a}$, then clearly $\mathfrak{q} \supset f(\mathfrak{a})$ whence $\mathfrak{q} \supset \mathfrak{a}B$, since $\mathfrak{a}B$ is the smallest ideal in B containing $f(\mathfrak{a})$. Conversely, if $\mathfrak{q} \supset \mathfrak{a}B$, then $\mathfrak{q} \supset f(\mathfrak{a})$, and hence $\mathfrak{q}^c = f^{-1}(\mathfrak{q}) \supset f^{-1}(f(\mathfrak{a})) \supset \mathfrak{a}$. Thus

$$\mathfrak{q}^c \in V(\mathfrak{a}) \iff \mathfrak{q} \in V(\mathfrak{a}B).$$

Another way of saying this is

$$({}^a f)^{-1}(V(\mathfrak{a})) = V(\mathfrak{a}B). \tag{1.2.1}$$

Incidentally, this is another proof that ${}^a f\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is continuous.

2. **Localization and relative topology**

We make one notational change, just for this section. Let $\phi\colon R \to R'$ be a ring map. Suppose S is a multiplicative system in R. We will write

$$\phi_S\colon S^{-1}R \longrightarrow S^{-1}R' \tag{†}$$

for the map $S^{-1}\phi$. This notation is typographically a tad more convenient for the story we spin in this section. Recall that if $T = \phi(S)$, then $S^{-1}R' = T^{-1}R'$, and so $\phi_S(x/s) = \phi(x)/s = \phi(x)/\phi(s)$.

2.1. **A homeomorphism.** Let A be a ring, S a multiplicative system in A. For A-module M, $i_M\colon M \to S^{-1}M$ will denote the localization map. In particular we have $i_A\colon A \to S^{-1}A$, which is more than a A-module map, since it is a ring homomorphism. Let $X = \operatorname{Spec}(A)$ and $X_S = \operatorname{Spec}(S^{-1}A)$. We then have a continuous map

$${}^a i_A\colon X_S \longrightarrow X. \tag{2.1.1}$$

Let $D(S) = \{\mathfrak{p} \in X \mid \mathfrak{p} \cap S = \emptyset\}$. We have seen (from Problem 6 of HW 1) that ${}^a i_A$ factors through $D(S)$, that it is injective, and its image is $D(S)$. In other words we have a commutative diagram

$$\begin{array}{ccc} X_S & \longrightarrow & D(S) \\ & {}^{{}^a i_A}\searrow & \Big\downarrow {\scriptstyle \text{inclusion}} \\ & & X \end{array} \tag{2.1.2}$$

with the horizontal arrow being bijective. The inverse of the horizontal arrow in (2.1.2) is $\mathfrak{p} \mapsto S^{-1}\mathfrak{p}$. We claim the horizontal arrow, namely

$$X_S \to D(S), \qquad \mathfrak{q} \mapsto {}^a i_A(\mathfrak{q}) = \mathfrak{q}^c, \tag{2.1.3}$$

is a homeomorphism, where $D(S)$ is given the subspace topology from X. Since (2.1.3) is continuous and bijective, we only have to show that it is an open map, which we now proceed to do.

It is enough for us to show that ${}^a i_A(D(g/s)) = D(g) \cap D(S)$ for $g \in A$ and $s \in S$. Since $s/1$ is a unit in $S^{-1}A$, $D(g/s) = D(g/1)$, and so it is enough to show that ${}^a i_A(D(g/1)) = D(g) \cap D(S)$. If $\mathfrak{q} \in D(g/1)$, and $\mathfrak{p} = {}^a i_A \mathfrak{q} = i_A^{-1}(\mathfrak{q})$, then $g \notin \mathfrak{p}$, whence $\mathfrak{p} \in D(g)$, i.e. $\mathfrak{p} \in D(S) \cap D(g)$. Conversely, if $\mathfrak{p} \in D(S)$ is such that $g/1 \in S^{-1}\mathfrak{p}$, then $g/1 = x/s$, for some $x \in \mathfrak{p}$, and $s \in S$. It follows that there exists $t \in S$ such that $tsg = tx$, i.e. $tsg \in \mathfrak{p}$. We therefore have $g \in \mathfrak{p}$, since $ts \notin \mathfrak{p}$. Thus $\mathfrak{p} \notin D(g)$. In other words, if $\mathfrak{p} \in D(g) \cap D(S)$, then $S^{-1}\mathfrak{p} \in D(g/1)$. This establishes our claim.

In main takeaway is that the map (2.1.3) is a homeomorphism. As a special case we see that if $f \in A$, then $\operatorname{Spec}(A_f)$ is homeomorphic to $D(f)$.

2.2. **The inverse image of $D(S)$.** Let A and S be as above. We retain the notations of §2.1. Let $f\colon A \to B$ be a ring homomorphism. Let $Y = \operatorname{Spec}(B)$. Consistent with our notation X_S for $\operatorname{Spec}(S^{-1}A)$, we write $Y_{f(S)}$ or Y_S for $\operatorname{Spec}(S^{-1}B)$. We have a continuous map ${}^af\colon Y \to X$ and a subset $D(f(S))$ of Y. Recall that $D(f(S))$ is the collection of prime ideals $\mathfrak{q}$ in B such that $\mathfrak{q} \cap f(S) = \emptyset$. We claim that

$$({}^af)^{-1}(D(S)) = D(f(S)). \tag{2.2.1}$$

This follows from the equivalence:

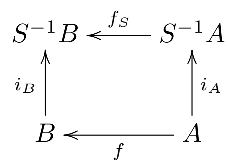
$$f(S) \cap \mathfrak{q} = \emptyset \quad \Longleftrightarrow \quad S \cap f^{-1}(\mathfrak{q}) = \emptyset. \tag{$*$}$$

The ($\Rightarrow$) direction of ($*$) follows from the inclusion

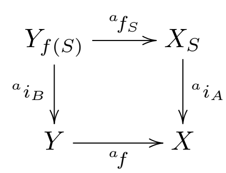
$$S \cap f^{-1}(\mathfrak{q}) \subset f^{-1}(f(S)) \cap f^{-1}(\mathfrak{q}) = f^{-1}(f(S) \cap \mathfrak{q}).$$

The ($\Leftarrow$) direction is seen as follows. Suppose $f(S) \cap \mathfrak{q} \neq \emptyset$. Then there exists $s \in S$ such that $f(s) \in \mathfrak{q}$. Thus $S \cap f^{-1}(\mathfrak{q})$ is a non empty set since s lies in it.

The relation (2.2.1) can (should?) be interpreted as follows: The commutative diagram

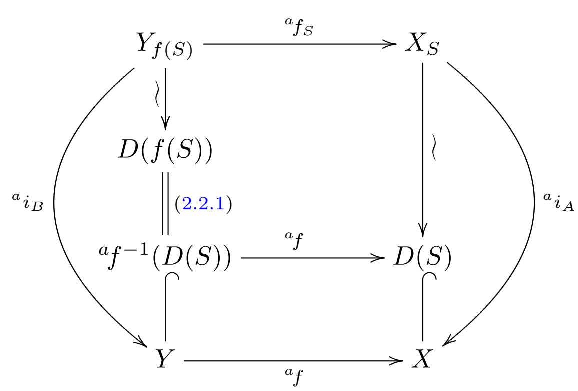


induces a commutative diagram



which, in view of (2.2.1), can be expanded to the following commutative diagram (with the isomorphisms being isomorphisms in the category of topological spaces, i.e. homeomorphisms):

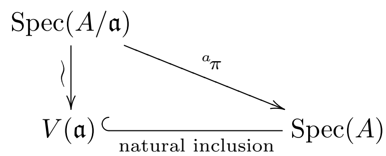
(2.2.2)



If we identify X_S with $D(S)$ and $Y_{f(S)}$ with $D(f(S))$, and regard ${}^ai_A$ and ${}^ai_B$ as the inclusions $D(S) \subset X$, and $D(f(S)) \subset Y$ respectively, then $({}^af)^{-1}(X_S) = Y_{f(S)}$ and the map ${}^af_S$ is the restriction of ${}^af$ to $Y_{f(S)}$. This is the interpretation of (2.2.1) that is most useful.

2.3. **The fibre ${}^af^{-1}(\mathfrak{p})$.** Let A be a ring and $\mathfrak{a}$ an ideal. The bijective correspondence between ideals of $A/\mathfrak{a}$ and ideals of A containing $\mathfrak{a}$ shows that $V(I)$ and $\operatorname{Spec}(A/\mathfrak{a})$ can be identified as topological spaces. More precisely, if $\pi\colon A \twoheadrightarrow A/\mathfrak{a}$ is the canonical surjection, then the map ${}^a\pi\colon \operatorname{Spec}(A/\mathfrak{a}) \to \operatorname{Spec}(A)$ is such that ${}^a\pi(\operatorname{Spec}(A/\mathfrak{a})) = V(\mathfrak{a})$ and the induced map $\operatorname{Spec}(A/\mathfrak{a}) \to V(\mathfrak{a})$ is a homeomorphism. In other words we have a commutative diagram

(2.3.1)



The downward pointing isomorphism (on the left) is an isomorphism in the category of topological spaces, i.e. it is a homeomorphism. One does not distinguish between $\operatorname{Spec}(A/\mathfrak{a})$ and $V(\mathfrak{a})$. This identification has uses as we will see below.

Now suppose $f\colon A \to B$ is a ring homomorphism, $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$.

Proposition 2.3.2. *The natural map* $\operatorname{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}) \to Y$ *induced by the ring homomorphism* $B \to B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ *maps* $\operatorname{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p})$ *homeomorphically on to the fibre* ${}^af^{-1}(\mathfrak{p})$.

Proof. First assume A is a local ring and $\mathfrak{p}$ is the maximal ideal of A. The Proposition follows (in this special case) from the commutative diagrams (1.2.1) and (2.3.1), since $B_\mathfrak{p} = B$ in this case.

In the general case let $S = A \smallsetminus \mathfrak{p}$. Identifying $D(S)$ with X_S and $D(f(S)$ with $Y_S = Y_{f(S)}$ with as we did in (2.1.3), we see from (2.2.2) that the fibre of ${}^af$ over $\mathfrak{p}$ can be identified with the fibre of ${}^af_S$ over $S^{-1}\mathfrak{p} = \mathfrak{p}A_\mathfrak{p}$. We are then done by the special case we considered above, since $X_S = \operatorname{Spec}(A_\mathfrak{p})$, $Y_S = \operatorname{Spec}(B_\mathfrak{p})$, f_S is the natural map $f_\mathfrak{p}\colon A_\mathfrak{p} \to B_\mathfrak{p}$, and $\mathfrak{p}A_\mathfrak{p}$ is the maximal ideal of the local ring $A_\mathfrak{p}$. □

2.3.3. A couple of observations are in order.

1. Let $\kappa(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$, the residue field of the local ring $A_\mathfrak{p}$. By (#), $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} = B_\mathfrak{p} \otimes_A A/\mathfrak{p} = B \otimes_A A_\mathfrak{p} \otimes_A A/\mathfrak{p} = B \otimes_A \kappa(\mathfrak{p})$. It is worth pointing out that since $B \otimes_A A_\mathfrak{p} \otimes_A A/\mathfrak{p} = B \otimes_A A/\mathfrak{p} \otimes_A A_\mathfrak{p}$, we have $B \otimes_A \kappa(\mathfrak{p}) = B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} = (B/\mathfrak{p}B)_\mathfrak{p}$.
   Proposition 2.3.2 is usually remembered by algebraic geometers via the formula

$$ {}^af^{-1}(\mathfrak{p}) = \operatorname{Spec}(B \otimes_A \kappa(\mathfrak{p})). \tag{2.3.3.1} $$

2. Most of the discussion so far in this lecture is an elaborate solution to Exercise 21 in Chapter 3 of [AM].

3. Integral extensions

3.1. Let A be a subring of a ring B. An element $x \in B$ is said to be *integral over* A if there exist an integer $n \geq 1$ and elements $a_1, \dots, a_n \in A$ such that

$$ x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0. \tag{3.1.1} $$

In other words x is a root of a monic polynomial

$$ X^n + a_1X^{n-1} + \cdots + a_{n-1}X + a_n \in A[X] \tag{3.1.2} $$

over A. The ring B is said to be an *integral extension over* A if every element of B is integral over A. We often abbreviate the phrase "B is an integral extension over A" to "B is integral over A". Alternately, we may say "$A \subset B$ is an integral extension".

3.2. **Toward the going up theorem.** Throughout this subsection, we will assume $A \subset B$ is an integral extension.

1. Let A and B be integral domains. Then A is a field if and only if B is a field.
Proof. Suppose A is a field. Let x be a non-zero element of B. Let polynomial (3.1.2) be a monic polynomial of least degree satisfied by x. Since B is an integral domain, $a_n \neq 0$, for otherwise $x^{n-1} + a_1x^{n-1} + \cdots + a_{n-2}x + a_{n-1} = 0$, violating the condition that the polynomial in (3.1.2) is a polynomial of minimal degree satisfied by x. Clearly

$$x(x^{n-1} + a_1x^{n-1} + \cdots + a_{n-2}x + a_{n-1}) = -a_n.$$

Since A is a field and $a_n \neq 0$, a_n is a unit in A and hence in B. In other words, x is a unit in B.

Conversely, suppose B is a field and let a be non-zero element of A. Let x be the multiplicative inverse of a in B. Assume again that x satisfies the equation (3.1.1) and that the associated polynomial in (3.1.2) is a monic polynomial of least degree for which x is a zero. Once again, clearly $a_n \neq 0$ because of the minimality condition. Now multiplying (3.1.1) by a^n and using the fact that $ax = 1$ we get

$$a\left\{-\sum_{k=1}^{n} a_k a^{k-1}\right\} = 1.$$

Since the expression in braces is an element in A, a is a unit in A. □

2. Let S be a multiplicative system in A. Then $S^{-1}A \subset S^{-1}B$ is also an integral extension. (We are using the fact that localization is exact to identify $S^{-1}A$ with a subring of $S^{-1}B$.)
Proof. Suppose $x/s \in S^{-1}B$, with $x \in B$ and $s \in S$. Assume x satisfies the integral relation (3.1.1). Consider the monic polynomial g in $(S^{-1}A)[X]$ given by

$$g = X^n + \frac{a_1}{s}X^{n-1} + \frac{a_2}{s^2}X^{n-2} + \cdots + \frac{a_{n-1}}{s^{n-1}}X + \frac{a_n}{s^n}.$$

It is clear that $g(x/s) = 0$, whence x/s is integral over $S^{-1}A$. □

3. Let $\mathfrak{b}$ be an ideal in B and $\mathfrak{a} = \mathfrak{b}^c$, so that the natural map $A/\mathfrak{a} \to B/\mathfrak{b}$ is injective. Regard $A/\mathfrak{a}$ as a subring of $B/\mathfrak{b}$ is this manner. Then $B/\mathfrak{b}$ is integral over A.
Proof. For $a \in A$, let $\bar{a} = a + \mathfrak{a} \in A/\mathfrak{a}$ and for $x \in B$, let $\bar{x} = x + \mathfrak{b} \in B/\mathfrak{b}$. If $x \in B$ satisfies (3.1.1), then it is clear that

$$\bar{x}^n + \bar{a}_1\bar{x}^{n-1} + \cdots + \bar{a}_{n-1}\bar{x} + \bar{a}_n = 0.$$

giving the required result. □

4. Suppose A is a local ring with maximal ideal $\mathfrak{m}$. A prime ideal $\mathfrak{q}$ in B is maximal if and only if $\mathfrak{q}^c = \mathfrak{m}$.
Proof. Suppose $\mathfrak{q}$ is a prime ideal in B such that $\mathfrak{q}^c = \mathfrak{m}$. Then $B/\mathfrak{q}$ is integral over $A/\mathfrak{m}$ by **3**. The rings $A/\mathfrak{m}$ and $B/\mathfrak{q}$ are integral domains. By **1**, $B/\mathfrak{q}$ is a field, i.e. $\mathfrak{q}$ is a maximal ideal of B.

Conversely, suppose $\mathfrak{q}$ is a maximial ideal of B and $\mathfrak{p} = \mathfrak{q}^c$. Then $B/\mathfrak{q}$ is integral over $A/\mathfrak{p}$. The rings $A/\mathfrak{p}$ and $B/\mathfrak{q}$ are both integral domains. Invoking **1** again we see that $\mathfrak{p}$ is a maximal ideal of A, and hence $\mathfrak{p} = \mathfrak{m}$. □

5. If A is a field then all prime ideals of B are maximal.
Proof. This is an immediate consequence of **4**. □

6. The map $\phi\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ induced by $A \subset B$ is surjective.
Proof. Let $\mathfrak{p}$ be a prime ideal of A. Let $\phi_{\mathfrak{p}}\colon \mathrm{Spec}(B_{\mathfrak{p}}) \to \mathrm{Spec}(A_{\mathfrak{p}})$ be the map induced by the integral extension $A_{\mathfrak{p}} \subset B_{\mathfrak{p}}$. By Proposition 2.3.2, or better still, by the commutative diagram (2.2.2), the fibre $\phi^{-1}(\mathfrak{p})$ can be identified with the fibre $\phi_{\mathfrak{p}}^{-1}(\mathfrak{p}A_{\mathfrak{p}})$. By **4**, every maximal ideal of $B_{\mathfrak{p}}$ lies in $\phi_{\mathfrak{p}}^{-1}(\mathfrak{p}A_{\mathfrak{p}})$, which means $\phi_{\mathfrak{p}}^{-1}(\mathfrak{p}A_{\mathfrak{p}})$, and hence $\phi^{-1}(\mathfrak{p})$, is non-empty. This shows that ϕ is surjective.

References

[AM] M.F. Atiyah and I.G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, London, 1969.