LECTURE 12

Date of Lecture: Feb 22, 2022

1. Irreducible components of Spec A

Let X be a topological space. Recall that X is said to be *irreducible* if $X = F_1$ or $X = F_2$ whenever $X = F_1 \cup F_2$, with F_1 and F_2 closed in X. A subset Z is said to be an irreducible subset if it is irreducible in the topology it inherits from X. If Z is irreducible, so is its topological closure $\overline{Z}$.

Recall further that a subset Z of X is an *irreducible component of* X if it is a maximal irreducible subset of X. Irreducible components are necessarily closed subsets (consider their closure).

We saw in the last lecture that every irreducible set is contained in an irreducible component and that X is the union of its irreducible components.

For the rest of this section, A is a ring and $X = \operatorname{Spec} A$.

1.1. **Reduced rings.** A is said to be *reduced* if it contains no non-zero nilpotent elements, i.e. if $x^n = 0$ for some $x \in A$, then $x = 0$. In other words, A is reduced if $\sqrt{0} = 0$, where $\sqrt{0}$ is the nilradical of A. In other words A is reduced if 0 is a radical ideal.

It is clear that I is a radical ideal of A if and only if A/I is reduced since the nilradical of A/I is clearly $\sqrt{I}/I$.

Since $\sqrt{I} = \{f \in A \mid f^n \in I \text{for some } n \geq 1\}$, it is clear that if I and J are radical ideals, then so is $I \cap J$.

We saw in Lecture 5 (with input from Lecture 4) that

$$\sqrt{I} = \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p}. \tag{1.1.1}$$

Some immediate consequences of (1.1.1) are:

- $V(I) = V(\sqrt{I})$.
- $\sqrt{I}$ is a radical ideal.
- If J is an ideal containing I, then $\sqrt{J/I} = \sqrt{J}/I$. Note that both sides are ideals in A/I.
- There is an inclusion reversing bijective correspondence between radical ideals and closed subsets of X, the correspondence being $I \longleftrightarrow V(I)$.

From these observations one deduces that the ring

$$A_{\text{red}} := A/\sqrt{0} \tag{1.1.1}$$

is a reduced ring. A_{red} is called the *reduced ring associated with* A. Since $V(I)$ can be identified as a topological space with $\operatorname{Spec}(A/I)$, therefore the relationship $V(0) = V(\sqrt{0})$ shows that $\operatorname{Spec} A$ and $\operatorname{Spec}(A_{\text{red}})$ can be identified. More precisely, if $\pi\colon A \to A_{\text{red}}$ is the natural surjection then the induced map of topological spaces, namely ${}^a\pi\colon \mathrm{Spec}(A_{\mathrm{red}}) \to \mathrm{Spec}\, A$, is a homeomorphism. Setting $X_{\mathrm{red}} = \mathrm{Spec}(A_{\mathrm{red}})$ we have the identification

$$X_{\mathrm{red}} = X. \tag{1.1.2}$$

Proposition 1.1.3. *X is irreducible if and only if A_{red} is an integral domain.*

Proof. Without loss of generality we may assume, by replacing A by A_{red} if necessary, that A is reduced. We have to show that X is irreducible if and only if A is an integral domain.

Suppose f and g are *non-zero* elements of A such that $fg = 0$. Since neither f nor g is a unit, $\langle f \rangle$ and $\langle g \rangle$ are proper ideals, whence X is not equal to either $V(\langle f \rangle)$ or $V(\langle g \rangle)$. We have

$$X = V(0) = V(\langle fg \rangle) = V(\langle f \rangle) \cup V(\langle g \rangle)$$

showing that X is not irreducible.

Conversely, suppose A is an integral domain. Suppose $X = F_1 \cup F_2$ where F_1 and F_2 are closed subsets of X. Then $F_1 = V(I)$ and $F_2 = V(J)$ where I and J are radical ideals. We have $X = V(I \cap J)$. Now $I \cap J$ is a radical ideal, and since A is an integral domain, so is 0. We have $V(0) = X = V(I \cap J)$, and by the bijective correspondence between radical ideals and closed subsets of X, we get $0 = I \cap J$. According to the very first Proposition in Lecture 8, this means that either I or J is equal to 0, since $\mathfrak{p} = 0$ is a prime ideal, A being an integral domain. This proves X is irreducible. □

Corollary 1.1.4. *A closed subset Z of X is irreducible if and only if $Z = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$ of A.*

Proof. Let $Z = V(I)$ where I is a radical ideal. We know that Z is homeomorphic to $\mathrm{Spec}(A/I)$, the map being $\mathfrak{p} \mapsto \mathfrak{p}/I$, $\mathfrak{p} \in V(I)$. Since A/I is reduced (I being a radical ideal), according to the Proposition, Z is irreducible if and only if A/I is an integral domain, i.e. if and only if I is a prime ideal. □

2. Noetherian topological spaces

2.1. Let X be a topological space. X is said to be *Noetherian* if it satisfies the descending chain condition for closed subsets, i.e. every descending chain of closed subsets

$$Z_0 \supset Z_1 \supset \cdots \supset Z_n \supset \dots \tag{2.1.1}$$

is stationary.

Remarks 2.1.2. The following facts are established by standard arguments.

- **(i)** If A is a Noetherian ring, then $\mathrm{Spec}\, A$ is a Noetherian topological space.
- **(ii)** X is a Noetherian topological space if and only if the open subsets of X satisfy the ascending chain condition for open sets in X; if and only if every non-empty collection of open sets has a maximal element (with respect to inclusion); if and only if every non-empty collection of closed subsets of X has a minimal element (with respect to inclusion).

Here is the main theorem concerning Noetherian topological spaces. The fact that a non-empty collection of closed sets has a minimal element plays a crucial role in the proof.

Theorem 2.1.3. (See [Ku, p.13, Proposition 2.14]) *Let X be a Noetherian topological space.*

(a) *X has only a finite number of irreducible components.*
(b) *No irreducible component of X is contained in the union of the remaining components.*

Proof. Let Σ be the collection of closed subsets of X which *cannot* be written as a finite union of irreducible sets. Suppose Σ is not empty. Then it has a minimal element Y. Now Y cannot be irreducible, for $Y \in \Sigma$. We can therefore find closed subsets Y_1, Y_2 of Y, with $Y_i \neq Y$, $i = 1, 2$ such that $Y = Y_1 \cup Y_2$. Now $Y_i \notin \Sigma$, $i = 1, 2$, and hence each Y_i can be written as a finite union of irreducible sets. It follows that Y can be written as a finite union of irreducible sets, contradicting the fact that $Y \in \Sigma$. Thus $\Sigma = \emptyset$. This means every closed set can be written as a finite union of irreducible sets.

Since X is closed, X too can be written as a finite union of irreducible sets. Since every irreducible set is contained in an irreducible component, X can in fact be written as finite union of irreducible components, say

$$X = \bigcup_{i=1}^{n} X_i$$

with X_i distinct irreducible components. We next establish that $X_1, \ldots, X_n$ are the only irreducible components of X. To that end, suppose Y is an irreducible component of X. Then

$$Y = \bigcup_{i=1}^{n} (Y \cap X_i).$$

Since Y is irreducible, this implies that $Y = Y \cap X_i$ for some i, i.e. $Y = X_i$.

Finally, suppose for some i, $X_i \subset \bigcup_{j \neq i} X_j$. Then

$$X_i = \bigcup_{j \neq i} (X_j \cap X_i)$$

whence $X_i = X_j \cap X_i$ for some $j \neq i$. This means $X_i = X_j$ for this j, contradicting the fact that the X_i are distinct. □

References

[Ku] E. Kunz *Introduction to Commutative Algebra and Algebraic Geometry*, Birkhaüser, Boston, Basel, Berlin, 1985.