Feb 15, 2022 Lecture 4 Alg II

Fix a ring A, and let $X = \operatorname{Spec} A$.
For a subset Z of X, let

$$I(Z) := \{ f \in A \mid f \in \mathfrak{p},\ \forall\ \mathfrak{p} \in Z \}.$$

Remark: $I(Z)$ is clearly an ideal.

Lemma: $\bar{Z} = V(I(Z))$ where $\bar{Z}$ denotes the closure Z in X.

Proof:

It is clear, by definition of $I(Z)$, that $Z \subseteq V(I(Z))$, and of hence $\bar{Z} \subset V(I(Z))$. Suppose now that

$$Z \subset V(J)$$

for some ideal J of I. If $f \in J$, then $f \in \mathfrak{p}$ $\forall\ \mathfrak{p} \in Z$, whence $f \in I(Z)$. Thus $J \subseteq I(Z)$, which in turn implies that $V(I(Z)) \subseteq V(J)$. Thus $\bar{Z} = V(I(Z))$ //

Further remarks:

Recall that for an ideal I, $\sqrt{I} := \{ f \in A \mid f^n \in I \text{ for some } n > 0 \}$, and

$$\sqrt{I} = \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p}.$$

It is clear that if I, J are ideals of A, then $V(I) = V(I)$ if and only if $\sqrt{I} = \sqrt{J}$. An ideal I is called a radical ideal if $I = \sqrt{I}$. From these remarks it is clear that there is a bijective correspondence between radical ideals and closed subsets of X, and the correspondence is s.t. $I \subset J \iff V(I) \supset V(J)$.

$D(f)$ and $\operatorname{Spec}(A_f)$:

For $f \in A$, let

$$i^f : A \longrightarrow A_f$$

denote the localization map ($i^f = i_A$ in our earlier notation, but now we wish to emphasize the role of f).

Let

$$X_f = \operatorname{Spec}(A_f)$$

and let

$$\phi_f : X_f \longrightarrow X$$

be the map induced by i^f. In other words, $\phi_f = {}^a i^f$.

According to Problem 6 of HW 1,

(a) $D(f) = \phi_f(X_f)$

(b) The map $X_f \longrightarrow D(f)$ induced by ϕ_f is bijective.

It is easy to verify that for $g \in A$

$$\phi_f(D(g/_1)) = D(f) \cap D(g) = D(fg).$$

Now for any $n \geqslant 0$, $D(g/_{f^n}) = D(g/_1)$, whence

$$\phi_f(D(g/_{f^n})) = D(fg).$$

We have therefore shown that ϕ_f is an open map. Since it is continuous and injective, and its image is $D(f)$, it follows that

$$X_f \xrightarrow[\phi_f]{} D(f)$$

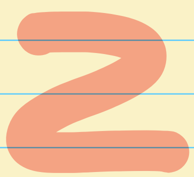
is a homeomorphism. The following commutative diagram summarizes the situation

$$\begin{tikzcd} X_f \arrow[r, hook, "\phi_f"] \arrow[dr, "\sim"'] & X \\ & D(f) \arrow[u, hook] \end{tikzcd}$$

homeomorphism

From now on, we identify $D(f)$ with X_f.

Quasi-compactness of $X = \operatorname{Spec} A$



The Bourbaki "dangerous bend" symbol.

A topological space (in this course and in courses in algebraic geometry) is called quasi-compact if every open cover of it has a finite subcover. We reserve the term compact for topological spaces which are Hausdorff in addition to being quasi-compact.

Lemma: Let $\{f_\gamma \in A \mid \gamma \in \Gamma\}$ be a collection of elements in A. Then $\langle f_\gamma \mid \gamma \in \Gamma \rangle = A$ if and only if $\bigcup_{\gamma \in \Gamma} D(f_\gamma) = X$.

Proof:

Let $I = \langle f_\gamma \mid \gamma \in \Gamma \rangle$. Then

$$\bigcup_{r\in\Gamma} D(f_r) = \bigcup_{r\in\Gamma} (X \setminus V(\langle f_r \rangle)) = X \setminus \bigcap_{r\in\Gamma} V(\langle f_r\rangle) = X\setminus V(I).$$

Thus $\bigcup_{r\in\Gamma} D(f_r) = X \iff V(I) = \phi \iff I = A$. //

Proposition: Let $f_r, r\in\Gamma$ be a collection of elements in A, and I the ideal generated by these elements. The following are equivalent:

(a) $I = A$

(b) $\exists$ positive integers $n_r, r\in\Gamma$ such that $\langle f_r^{n_r} \mid r\in\Gamma\rangle = A$

(c) For every choice of a family of positive integers $n_r, r\in\Gamma$, $\langle f_r^{n_r} \mid r\in\Gamma\rangle = A$.

Proof:

This follows from the simple observation that if $n_r > 0$ then

$$D(f_r^{n_r}) = D(f_r).$$ //

Proposition: X is quasi-compact.

Proof:

It is clearly enough to work with standard open sets. So suppose $\mathcal{U} = \{D(f_r) \mid r\in\Gamma\}$ is an open cover of X. Let $I = \langle f_r \mid r\in\Gamma\rangle$. We know from the above results that $I = A$, whence $1\in I$. In particular, there exist $r_1, r_2, \dots, r_n \in \Gamma$ and $a_1, a_2, \dots, a_n \in A$ such that

$$a_1 f_{r_1} + \dots + a_n f_{r_n} = 1.$$

It follows that $\langle f_{r_1}, \dots, f_{r_n}\rangle = A$, whence $X = \bigcup_{i=1}^{n} D(f_{\alpha_i})$. //

Out of turn. Should have done this earlier.

Closed points and maximal ideals

This should have been done right after proving $\bar{Z} = V(I(Z))$. Let $\mathfrak{p}$ be a prime ideal of A. Clearly (specializing to $Z = \{\mathfrak{p}\}$)

$$I(\{\mathfrak{p}\}) = \mathfrak{p}.$$

Thus

$$\overline{\{\mathfrak{p}\}} = V(I(\{\mathfrak{p}\})) = V(\mathfrak{p}).$$

It follows that $\{\mathfrak{p}\}$ is closed if and only if $V(\mathfrak{p}) = \{\mathfrak{p}\}$, i.e. if and only if $\mathfrak{p}$ is a maximal ideal.

Thus the closed points of X are precisely the maximal ideals of A.

Kunz's book "Introduction to Commutative Algebra and Algebraic Geometry" is a good reference for this section.

Irredicible subsets and irreducible components of a topological space

Most of the concepts here are mainly used in non-Hausdorff situations. The spectrum of a ring is rarely Hausdorff since in a Hausdorff space every point is a closed point, and from what we have seen above, a necessary condition then for Spec A to be Hausdorff is that every prime ideal should be a maximal ideal. In particular, if A is an integral domain Spec A is a Hausdorff if and only if A is a field, since $\langle 0 \rangle$ is a prime ideal in an integral domain.

Definition: A topological space X is called irreducible if for any decomposition $X = F_1 \cup F_2$ with F_1, F_2 closed, we have $X = F_1$ or $X = F_2$. A Z is called irreducible if it is irreducible in the induced topology, i.e. in its subspace topology.

Lemma: Let X be a topological space. The following are equivalent.

(a) X is irreducible

(b) If U_1, U_2 are two non-empty open subsets of X then $U_1 \cap U_2 \neq \phi$.

(c) Every non-empty open subset of X is dense in X

Proof:

(a) $\iff$ (b) Follows by taking complements of U_1 and U_2.

(b) $\iff$ (c) Follows from the fact that subset of X is dence if and only if it intersects every non-empty open set. //

Corollary: Let Z be a subset of X. TFAE

(a) Z is irreducible

(b) If U_i, $i=1,2$ are open in X with $U_i \cap Z \neq \phi$, $i=1,2$, then $Z \cap U_1 \cap U_2 \neq \phi$.

(c) The closure $\bar{Z}$ of Z is irreducible.

Proof:

For (c) the only thing to note is that for U open in X, $U \cap Z \neq \phi$ if and only if $U \cap \bar{Z} \neq \phi$. //

Definition: An irreducible component of a topological space X is a maximal irreducible subset of X.

Proposition: (a) Irreducible components are closed.

(b) Every irreducible subset of a topological space is contained in an irreducible component.

(c) Every topological space is the union of its irreducible components.

Proof:

(a) This follows from (c) of the previous Corollary.

(b) Let Z be an irreducible subset of X, and Σ_i the collection of irreducible subsets of X containing Z. Let $\{Z_\lambda \mid \lambda \in \Lambda\} \subset \Sigma_i$ be a totally ordered subset of Σ_i and set

$$Z_{max} := \bigcup_{\lambda \in \Lambda} Z_\lambda .$$

Suppose U_1, U_2 are open in X with $U_i \cap Z_{max} \neq \emptyset$, $i=1,2$. Then for some $\lambda \in \Lambda$, $Z_\lambda \cap U_i \neq \emptyset$ for $i=1,2$. Since Z_λ is irreducible, $Z_\lambda \cap U_1 \cap U_2 \neq \emptyset$, whence $Z_{max} \cap U_1 \cap U_2 \neq \emptyset$. Thus Z_{max} is irreducible. By Zorn's Lemma, Σ_i has a maximal element. This has to be an irreducible component.

(c) Follows from (b) since every point is irreducible. //