

Feb 10, 2022

Lecture 10

Alg II

Throughout this lecture, A is a ring.

Hom- \otimes adjointness

Here is the most basic form of the Hom- \otimes adjointness relation. There are fancier versions, one of which has been given to you as a problem in homework 3.

Theorem: Let $M, N, T \in \text{Mod}_A$. Then we have an isomorphism

$$\text{Hom}_A(M \otimes_A N, T) \xrightarrow{\sim} \text{Hom}_A(M, \text{Hom}_A(N, T)).$$

Proof:

Suppose $\phi \in \text{Hom}_A(M \otimes_A N, T)$. Define

$$\psi = \psi_\phi: M \longrightarrow \text{Hom}_A(N, T)$$

by the rule

$$(\psi(m))(n) = \phi(m \otimes n).$$

①

If $\phi_1, \phi_2 \in \text{Hom}_A(M \otimes_A N, T)$, and $a \in A$, then

$$\begin{aligned} ((\psi_{\phi_1} + a \psi_{\phi_2})(m))(n) &= (\psi_{\phi_1}(m))(n) + (\psi_{\phi_2}(am))(n) \\ &= \phi_1(m \otimes n) + \phi_2(am \otimes n) \\ &= \phi_1(m \otimes n) + (a \phi_2)(m \otimes n) \\ &= (\phi_1 + a \phi_2)(m \otimes n) \\ &= (\psi_{\phi_1 + a \phi_2}(m))(n) \end{aligned}$$

$\forall m \in M$
and $n \in N$.

$$\text{i.e., } \psi_{\phi_1} + a \psi_{\phi_2} = \psi_{\phi_1 + a \phi_2}.$$

Thus $\phi \mapsto \psi$ is an A -linear map.

We also have a map in the reverse direction

$\text{Hom}_A(M, \text{Hom}_A(N, T)) \longrightarrow \text{Hom}_A(M \otimes_A N, T)$,
described as follows:

Given $\psi \in \text{Hom}_A(M, \text{Hom}_A(N, T))$ we have a set theoretic map.

$$B_\psi: M \times N \longrightarrow T$$

given by

$$B_\psi(m, n) = (\psi(m))(n).$$

If $m_1, m_2 \in M$, $n_1, n_2 \in N$, and $a \in A$ we have

$$\begin{aligned} B_\psi(m_1 + m_2, n_1) &= (\psi(m_1 + m_2))(n_1) \\ &= \psi(m_1)(n_1) + \psi(m_2)(n_1) \\ &= B_\psi(m_1, n_1) + B_\psi(m_2, n_1). \end{aligned}$$

Similarly

$$B_\psi(m_1, n_1 + n_2) = B_\psi(m_1, n_1) + B_\psi(m_1, n_2).$$

Finally

$$\begin{aligned} B_\psi(am_1, n_1) &= \psi(am_1)(n_1) \\ &= ((a\psi)(m_1))(n_1) \\ &= \psi(m_1)(an_1) \\ &= B_\psi(m_1, an_1) \end{aligned}$$

and the second equality in the above chain gives

$$B_\psi(am_1, n_1) = ((a\psi)(m_1))(n_1) = a B_\psi(m_1, n_1).$$

Thus B_ψ is bilinear over A . This gives us a map

$$\phi: M \otimes_A N \longrightarrow T$$

with

$$\phi(m \otimes n) = B_\psi(m, n) = (\psi(m))(n) \quad \text{--- (2)}$$

It is not hard to see that $\psi \mapsto \phi$ is A -linear. The equations (1) and (2) above show that the two processes $\phi \mapsto \psi$ and $\psi \mapsto \phi$ are inverses of each other. //

Test for exactness

A sequence of A -maps

$$\dots \longrightarrow C^{i-1} \longrightarrow C^i \longrightarrow C^{i+1} \longrightarrow \dots$$

is said to be exact at i if $\text{im}(C^{i-1} \longrightarrow C^i) = \ker(C^i \longrightarrow C^{i+1})$.

If

$$M' \xrightarrow{\phi} M \xrightarrow{\psi} M''$$

is a pair of A -maps then we say the sequence is exact if

$$\text{im } \phi = \ker \psi.$$

In general given a sequence of A -maps, we say it is exact if it is exact at every place where there is an incoming and an outgoing arrow.

Examples:

1. A sequence of A -maps

$$M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is exact if (a) $\text{im}(M \longrightarrow M'') = \ker(M'' \longrightarrow 0)$ and (b) $\text{im}(M' \longrightarrow M) = \ker(M \longrightarrow M'')$. Condition (a) is equivalent to saying that $\text{im}(M \longrightarrow M'') = M''$, i.e. to saying that $M \longrightarrow M''$ is surjective. In view of this, condition (b) is equivalent to saying that M'' is essentially the cokernel of the map $M' \longrightarrow M$. More precisely we have a commutative diagram

$$\begin{array}{ccc} M & \longrightarrow & M'' \\ \parallel & & \uparrow \cong \leftarrow \text{natural isomorphism} \\ M & \longrightarrow & M / \ker(M \longrightarrow M'') \\ \parallel & & \parallel \\ M & \xrightarrow{\text{nat'l quotient map}} & M / \text{im}(M' \longrightarrow M) \end{array}$$

2. A sequence of maps

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M''$$

is exact if (a) $\text{im}(0 \longrightarrow M') = \ker(M' \longrightarrow M)$ and (b) $\text{im}(M' \longrightarrow M) = \ker(M \longrightarrow M'')$. Condition (a) is equivalent to saying that $M' \longrightarrow M$ is injective and in view of this (b) amounts to saying M' is essentially the kernel of $M \longrightarrow M''$. In greater detail, the diagram below commutes

$$\begin{array}{ccc} \ker(M \longrightarrow M'') & \subseteq & M \\ \text{natural isomorphism} \nearrow \cong & & \parallel \\ M' & \longrightarrow & M \end{array}$$

Some "functors"

Let $M, N, T \in \text{Mod}_A$.

1. $\text{Hom}_A(M, -)$: Suppose $\phi: N \rightarrow T$ is an A -map. Define a map

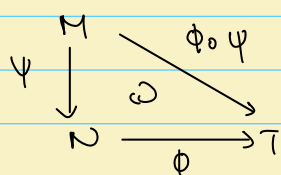
$$\text{Hom}_A(M, \phi): \text{Hom}_A(M, N) \longrightarrow \text{Hom}_A(M, T)$$

by the rule

$$\text{Hom}_A(M, \phi)(\psi)(m) = \phi \circ \psi(m).$$

It is simpler to write ϕ_* for $\text{Hom}_A(M, \phi)$.
Thus

$$\phi_*(\psi) = \phi \circ \psi$$



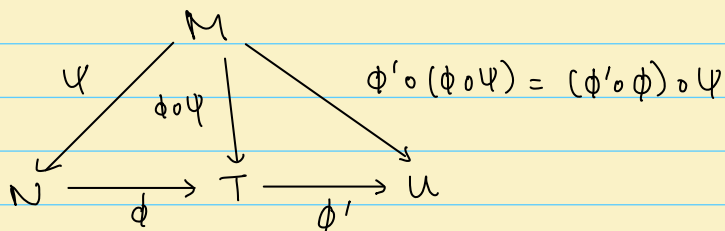
If $T \xrightarrow{\phi'} U$ is a second A -map, then is easy to see that

$$\text{Hom}_A(M, \phi') \circ \text{Hom}_A(M, \phi) = \text{Hom}_A(M, \phi' \circ \phi)$$

i.e.

$$\phi'_* \circ \phi_* = (\phi' \circ \phi)_*$$

This relationship is what makes $\text{Hom}_A(M, -)$ a "functor", a term we will define later in the course.



Check that ϕ_* is an A -map. (Easy and left to you.)

2. $\text{Hom}_A(-, N)$: Suppose $M \xrightarrow{\phi} T$ is an A -map. We have an obvious map

$$\phi^*: \text{Hom}_A(T, N) \longrightarrow \text{Hom}_A(M, N)$$

given by

$$\phi^*(\psi) = \psi \circ \phi, \quad \psi \in \text{Hom}_A(T, N).$$

The map ϕ^* is more accurately $\text{Hom}_A(\phi, N)$.

$$\text{Hom}_A(\phi, N) := \phi^*.$$

Observe that $\text{Hom}_A(-, N)$ reverses the direction of arrows. It however respects compositions, in the following sense:

If $\phi': T \rightarrow U$ is a second map, then

$$\phi^* \circ (\phi')^* = (\phi' \circ \phi)^*$$

The above relationship (i.e. $\text{Hom}_A(\phi, N) \circ \text{Hom}_A(\phi', N) = \text{Hom}_A(\phi' \circ \phi, N)$) is what makes $\text{Hom}_A(-, N)$ a functor (reminder: we still have to define the notion of a functor). More accurately, since $\text{Hom}_A(-, N)$ reverses the direction of arrows, it is a contravariant functor.

Finally, it is easy to check that ϕ^* is an A -map. The details are left to you.

3. $-\otimes_A N$ and $M \otimes_A -$: In much the same way $-\otimes_A N$ and $M \otimes_A -$ are functors. You can work out what is meant by $\phi \otimes_A N$ for an A -map $\phi: M \rightarrow T$, check that it respects compositions of A -maps, and that it is an A -map.

Left exactness of the Hom functors

The following properties of $\text{Hom}_A(M, -)$ and $\text{Hom}_A(-, M)$ are usually referred to as the left exactness of $\text{Hom}_A(M, -)$ and $\text{Hom}_A(-, M)$. We will define the notion of left exactness and right exactness (and exactness) more formally later in the course.

Theorem: Let $M \in \text{Mod}_A$.

(a) Suppose

$$0 \longrightarrow N' \xrightarrow{\phi} N \xrightarrow{\psi} N''$$

is an exact sequence of A -modules. Then the induced sequence

$$0 \longrightarrow \text{Hom}_A(M, N') \xrightarrow{\phi_*} \text{Hom}_A(M, N) \xrightarrow{\psi_*} \text{Hom}_A(M, N'')$$

is also exact.

(b) Suppose

$$N' \xrightarrow{\phi} N \xrightarrow{\psi} N'' \longrightarrow 0$$

is an exact sequence of A -modules. Then the sequence

$$0 \longrightarrow \text{Hom}_A(N'', M) \xrightarrow{\psi^*} \text{Hom}_A(N, M) \xrightarrow{\phi^*} \text{Hom}_A(N', M)$$

is also exact.

Proof:

(a) We have to show that ϕ_* is injective and that $\text{im}(\phi_*) = \ker(\psi_*)$. Suppose $f \in \text{Hom}_A(N', M)$ is s.t. $\phi_*(f) = 0$. Since $\phi_*(f) = \phi \circ f$, this means

$$\phi(f(m)) = 0 \quad \forall m \in M.$$

Now ϕ is injective, and hence the above equation implies that $f(m) = 0 \quad \forall m \in M$, i.e. $f = 0$. Thus ϕ_* is injective.

Now $\psi_* \circ \phi_* = (\psi \circ \phi)_* = 0$ since $\psi \circ \phi = 0$. It follows that $\text{im} \phi_* \subseteq \ker \psi_*$. We now show that $\ker \psi_* \subseteq \text{im} \phi_*$. Let $f \in \ker \psi_* \subseteq \text{Hom}_A(M, N)$. Then $\psi \circ f = \psi_* f = 0$, i.e.

$$\psi(f(m)) = 0 \quad \forall m \in M.$$

Hence $f(m) \in \ker \psi$ for every $m \in M$. Since

$$0 \longrightarrow N' \xrightarrow{\phi} N \xrightarrow{\psi} N'' \text{ is exact, } \phi(N') = \ker \psi, \text{ and}$$

the map $x \mapsto \phi(x)$ gives an isomorphism

$$\bar{\phi}: N' \xrightarrow{\sim} \phi(N').$$

Since f takes values in $\ker \psi = \phi(N')$, if we define

$$g: M \longrightarrow N'$$

by the formula

$$g(x) = \bar{\phi}^{-1}(f(x)), \quad x \in M,$$

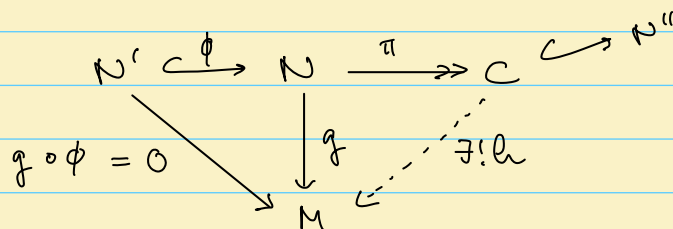
then $\phi \circ g = f$, i.e. $\phi_*(g) = f$. Thus $f \in \text{im} \phi_*$.

This proves (a).

Remark: The above argument can be re-packaged invoking the universal properties of kernels, namely, if $K = \ker(\psi)$ and $f: M \longrightarrow N$ is an A -map s.t. $\psi \circ f = 0$, then $\exists!$ map $g: M \longrightarrow K$ s.t. $f = i \circ g$, where $i: K \longrightarrow N$ is the inclusion map $K \subseteq N$. Since

$$\begin{array}{ccc} K & \xrightarrow{i} & N \\ \uparrow \bar{i} & & \parallel \\ N' & \xrightarrow{\phi} & N \end{array} \quad \begin{array}{l} \text{commutes,} \\ \text{we are done.} \end{array}$$

(b) The argument is dual to the one given in (a). Since $\psi: N \twoheadrightarrow N''$ is surjective, if $f \circ \psi = 0$ for an A -map $f: N'' \rightarrow M$ then $f = 0$. This means ψ^* is injective. For the remaining part, use the universal property of cokernels, namely if $C = \text{coker}(\phi)$, $\pi: N \twoheadrightarrow C = N/(\text{im}(\phi))$ the natural surjective map, and $g: N \rightarrow M$ an A -map such that $g \circ \phi = 0$, then $\exists!$ map $h: C \rightarrow M$ such that $g = h \circ \pi$.



The details are left to you. The diagram on the left may help.

The right exactness of $-\otimes_A N$

We need the following lemma

Lemma: Suppose we have a pair of A -maps

$$M \xrightarrow{\phi} N \xrightarrow{\psi} T$$

Then the sequence is exact (i.e. $\text{im} \phi = \ker \psi$) if $\text{Hom}_A(T, P) \xrightarrow{\psi^*} \text{Hom}_A(N, P) \xrightarrow{\phi^*} \text{Hom}_A(M, P)$ is an exact sequence of A -module for every $P \in \text{Mod}_A$.

Proof:

Take $P = T$. Then $\psi = \psi^*(1_P) \in \text{im} \psi^* = \ker \phi^*$, i.e. $\psi^*(\psi) = 0$, i.e. $\psi \circ \phi = 0$. This means $\text{im} \phi \subseteq \ker \psi$.

Next take $P = \text{coker} \phi$. Let $\pi \in \text{Hom}_A(N, P)$ be the natural map $N \twoheadrightarrow \text{coker} \phi = P$. Now $\phi^*(\pi) = \pi \circ \phi = 0$. Thus $\pi \in \ker(\phi^*) = \text{im}(\psi^*)$. This means $\exists h: T \rightarrow \text{coker} \phi = P$ such that $h \circ \psi = \pi$. Thus if $n \in \ker \psi \subseteq N$, then $\pi(n) = (h \circ \psi)(n) = h(\psi(n)) = 0$. It follows that $n \in \text{im} \phi$. Thus $\ker \psi \subseteq \text{im} \phi$. //

We are now in a position to prove the right exactness of $-\otimes_A N$ (see next page).

Theorem: Let

$$N' \xrightarrow{\phi} N \xrightarrow{\psi} N'' \longrightarrow 0$$

be an exact sequence of A -modules. Then for any $M \in \text{Mod}_A$ the induced sequence

$$M \otimes_A N' \xrightarrow{M \otimes_A \phi} M \otimes_A N \xrightarrow{M \otimes_A \psi} M \otimes_A N'' \longrightarrow 0$$

is exact.

Proof: Let $P \in \text{Mod}_A$. Then

$$0 \longrightarrow \text{Hom}_A(N'', P) \xrightarrow{\psi^*} \text{Hom}_A(N, P) \xrightarrow{\phi^*} \text{Hom}_A(N', P)$$

is exact. This induces an exact sequence

$$0 \longrightarrow \text{Hom}_A(M, \text{Hom}_A(N'', P)) \xrightarrow{\text{via } \psi} \text{Hom}_A(M, \text{Hom}_A(N, P)) \xrightarrow{\text{via } \phi} \text{Hom}_A(M, \text{Hom}_A(N', P))$$

By the $\text{Hom}-\otimes$ adjointness, this gives an exact sequence

$$0 \longrightarrow \text{Hom}_A(M \otimes_A N'', P) \xrightarrow{(M \otimes_A \psi)^*} \text{Hom}_A(M \otimes_A N, P) \xrightarrow{(M \otimes_A \phi)^*} \text{Hom}_A(M \otimes_A N', P).$$

The theorem follows from the previous lemma. //