Throughout this lecture, A is a ring.

Hom-$\otimes$ adjointness

Here is the most basic form of the Hom-$\otimes$ adjointness relation. There are fancier versions, one of which has been given to you as a problem in Homework 3.

Theorem: Let $M, N, T \in \mathrm{Mod}_A$. Then we have an isomorphism

$$\mathrm{Hom}_A(M \otimes_A N, T) \xrightarrow{\sim} \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, T)).$$

Proof:

Suppose $\phi \in \mathrm{Hom}_A(M \otimes_A N, T)$. Define

$$\psi = \psi_\phi : M \longrightarrow \mathrm{Hom}_A(N, T)$$

by the rule

$$(\psi(m))(n) = \phi(m \otimes n). \qquad \text{①}$$

If $\phi_1, \phi_2 \in \mathrm{Hom}_A(M \otimes_A N, T)$, and $a \in A$, then

$$\left.\begin{aligned}
((\psi_{\phi_1} + a\psi_{\phi_2})(m))(n) &= (\psi_{\phi_1}(m))(n) + (\psi_{\phi_2}(am))(n) \\
&= \phi_1(m \otimes n) + \phi_2((am) \otimes n) \\
&= \phi_1(m \otimes n) + (a\phi_2)(m \otimes n) \\
&= (\phi_1 + a\phi_2)(m \otimes n) \\
&= (\psi_{\phi_1 + a\phi_2}(m))(n)
\end{aligned}\right\} \forall m \in M \text{ and } n \in N.$$

i.e., $\psi_{\phi_1} + a\psi_{\phi_2} = \psi_{\phi_1 + a\phi_2}$.

Thus $\phi \longmapsto \psi$ is an A-linear map.

We also have a map in the reverse direction

$$\mathrm{Hom}_A(M, \mathrm{Hom}_A(N, T)) \longrightarrow \mathrm{Hom}_A(M \otimes_A N, T),$$

described as follows:

Given $\Psi \in \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, T))$ we have a set theoretic map.

$$B_\Psi : M \times N \longrightarrow T$$

given by

$$B_\Psi(m,n) = (\Psi(m))(n).$$

If $m_1, m_2 \in M$, $n_1, n_2 \in N$, and $a \in A$ we have

$$\begin{aligned} B_\Psi(m_1 + m_2, n_1) &= (\Psi(m_1+m_2))(n_1) \\ &= \Psi(m_1)(n_1) + \Psi(m_2)(n_1) \\ &= B_\Psi(m_1, n_1) + B_\Psi(m_2, n_1). \end{aligned}$$

Similarly

$$B_\Psi(m_1, n_1+n_2) = B_\Psi(m_1, n_1) + B_\Psi(m_1, n_2).$$

Finally

$$\begin{aligned} B_\Psi(am_1, n_1) &= \Psi(am_1)(n_1) \\ &= ((a\Psi)(m_1))(n_1) \\ &= \Psi(m_1)(an_1) \\ &= B_\Psi(m_1, an_1) \end{aligned}$$

and the second equality in the above chain gives

$$B_\Psi(am_1, n_1) = ((a\Psi)(m_1))(n_1) = a\, B_\Psi(m_1, n_1).$$

Thus B_Ψ is bilinear over A. This gives us a map

$$\phi : M \otimes_A N \longrightarrow T$$

with

$$\phi(m \otimes n) = B_\Psi(m,n) = (\Psi(m))(n) \qquad \text{——— } 2$$

It is not hard to see that $\Psi \longmapsto \phi$ is A-linear. The equations 1 and 2 above show that the two processes $\phi \longmapsto \Psi$ and $\Psi \longmapsto \phi$ are inverses of each other. //

Test for exactness

A sequence of A-maps

$$\cdots \longrightarrow C^{i-1} \longrightarrow C^i \longrightarrow C^{i+1} \longrightarrow \cdots$$

is said to be exact at i if $\mathrm{im}(C^{i-1} \longrightarrow C^i) = \ker(C^i \longrightarrow C^{i+1})$.

If

$$M' \xrightarrow{\phi} M \xrightarrow{\Psi} M''$$

is a pair of A-maps then we say the sequence is exact if

$\operatorname{im}\phi = \ker\psi$.

In general given a sequence of A-maps, we say it is exact if it is exact at every place where there is an incoming and an outgoing arrow.

Examples:

1. A sequence of A-maps
$$M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$
is exact if (a) $\operatorname{im}(M \longrightarrow M'') = \ker(M'' \longrightarrow 0)$ and (b) $\operatorname{im}(M' \longrightarrow M) = \ker(M \longrightarrow M'')$. Condition (a) is equivalent to saying that $\operatorname{im}(M \longrightarrow M'') = M''$, i.e. to saying that $M \longrightarrow M''$ is surjective. In view of this, condition (b) is equivalent to saying that M'' is essentially the cokernel of the map $M' \longrightarrow M$. More precisely we have a commutative diagram
$$\begin{tikzcd} M \arrow[r, two heads] & M'' \\ M \arrow[u, equal] \arrow[r] & M/\ker(M \longrightarrow M'') \arrow[u, "\wr"'] \\ M \arrow[u, equal] \arrow[r, "\text{nat'l quotient map}"'] & M/\operatorname{im}(M' \longrightarrow M) \arrow[u, equal] \end{tikzcd}$$
← natural isomorphism.

2. A sequence of maps
$$0 \longrightarrow M' \longrightarrow M \longrightarrow M''$$
is exact if (a) $\operatorname{im}(0 \longrightarrow M') = \ker(M' \longrightarrow M)$ and (b) $\operatorname{im}(M' \longrightarrow M) = \ker(M \longrightarrow M'')$. Condition (a) is equivalent to saying that $M' \longrightarrow M$ is injective and in view of this (b) amounts to saying M' is essentially the kernel of $M \longrightarrow M''$. In greater detail, the diagram below commutes
$$\begin{tikzcd} \ker(M \longrightarrow M'') \arrow[r, phantom, "\subseteq"] & M \\ M' \arrow[u, "\wr"] \arrow[r] & M \arrow[u, equal] \end{tikzcd}$$
natural isomorphism →

Some "functors"

Let $M, N, T \in Mod_A$.

1. $Hom_A(M, -)$: Suppose $\phi : N \longrightarrow T$ is an A-map. Define a map

$$Hom_A(M, \phi) : Hom_A(M, N) \longrightarrow Hom_A(M, T)$$

by the rule

$$\boxed{Hom_A(M, \phi)(\psi)(m) = \phi \circ \psi.}$$

$$\begin{tikzcd} M \arrow[d, "\psi"'] \arrow[dr, "\phi\circ\psi"] & \\ N \arrow[r, "\phi"'] \arrow[ur, phantom, "\circlearrowright"] & T \end{tikzcd}$$

It is simpler to write ϕ_* for $Hom_A(M, \phi)$. Thus

$$\boxed{\phi_*(\psi) = \phi \circ \psi}$$

If $T \xrightarrow{\phi'} U$ is a second A-map, then is easy to see that

$$\boxed{\begin{gathered} Hom_A(M, \phi') \circ Hom_A(M, \phi) = Hom_A(M, \phi' \circ \phi) \\ \text{i.e.} \\ \phi'_* \circ \phi_* = (\phi' \circ \phi)_* \end{gathered}}$$

This relationship is what makes $Hom_A(M, -)$ a "functor", a term we will define later in the course.

$$\begin{tikzcd} & M \arrow[dl, "\psi"'] \arrow[d, "\phi\circ\psi"] \arrow[dr] & \\ N \arrow[r, "\phi"'] & T \arrow[r, "\phi'"'] & U \end{tikzcd} \qquad \phi' \circ (\phi \circ \psi) = (\phi' \circ \phi) \circ \psi$$

Check that ϕ_* is an A-map. (Easy and left to you.)

2. $Hom_A(-, N)$: Suppose $M \xrightarrow{\phi} T$ is an A-map. We have an obvious map

$$\phi^* : Hom_A(T, N) \longrightarrow Hom_A(M, N)$$

given by

$$\phi^*(\psi) = \psi \circ \phi \quad , \quad \psi \in Hom_A(T, N).$$

The map ϕ^* is more accurately $Hom_A(\phi, N)$.

$$Hom_A(\phi, N) := \phi^*.$$

Observe that $\mathrm{Hom}_A(-,N)$ reverses the direction of arrows. It however respects compositions, in the following sense:
If $\phi' : T \longrightarrow U$ is a second map, then

$$\boxed{\phi^* \circ (\phi')^* = (\phi' \circ \phi)^*}$$

The above relationship (i.e. $\mathrm{Hom}_A(\phi, N) \circ \mathrm{Hom}_A(\phi', N) = \mathrm{Hom}_A(\phi' \circ \phi, N)$) is what makes $\mathrm{Hom}_A(-,N)$ a functor (reminder: we still have to define the notion of a functor). More accurately, since $\mathrm{Hom}_A(-,N)$ reverses the direction of arrows, it is a <u>contravariant functor</u>.

Finally, it is easy to check that <u>ϕ^* is an A-map</u>. The details are left to you.

3. <u>$-\otimes_A N$ and $M \otimes_A -$</u> : In much the same way $-\otimes_A N$ and $M \otimes_A -$ are functors. You can work out what is meant by $\phi \otimes_A N$ for an A-map $\phi : M \longrightarrow T$, check that it respects compositions of A-maps, and that it is an A-map.

<u>Left exactness of the Hom functors</u>

The following properties of $\mathrm{Hom}_A(M,-)$ and $\mathrm{Hom}_A(-,M)$ are usually referred to as the <u>left exactness</u> of $\mathrm{Hom}_A(M,-)$ and $\mathrm{Hom}_A(-,M)$. We will define the notion of left exactness and right exactness (and exactness) more formally later in the course.

<u>**Theorem:**</u> Let $M \in \mathrm{Mod}_A$.

(a) Suppose

$$0 \longrightarrow N' \xrightarrow{\phi} N \xrightarrow{\psi} N''$$

is an exact sequence of A-modules. Then the induced sequence

$$0 \longrightarrow \mathrm{Hom}_A(M,N') \xrightarrow[\phi_*]{} \mathrm{Hom}_A(M,N) \xrightarrow[\psi_*]{} \mathrm{Hom}_A(M,N'')$$

is also exact.

(b) Suppose
$$N' \xrightarrow{\phi} N \xrightarrow{\psi} N'' \longrightarrow 0$$
is an exact sequence of A-modules. Then the sequence
$$0 \longrightarrow \mathrm{Hom}_A(N'', M) \xrightarrow{\psi^*} \mathrm{Hom}_A(N, M) \xrightarrow{\phi^*} \mathrm{Hom}_A(N', M)$$
is also exact.

Proof:

(a) We have to show that ϕ_* is injective and that $\mathrm{im}(\phi_*) = \ker(\psi_*)$. Suppose $f \in \mathrm{Hom}_A(N', M)$ is s.t. $\phi_*(f) = 0$. Since $\phi_*(f) = \phi \circ f$, this means
$$\phi(f(m)) = 0 \quad \forall\, m \in M.$$
Now ϕ is injective, and hence the above equation implies that $f(m) = 0$ $\forall\, m \in M$, i.e. $f = 0$. Thus ϕ_* is injective.

Now $\psi_* \circ \phi_* = (\psi \circ \phi)_* = 0$ since $\psi \circ \phi = 0$. It follows that $\mathrm{im}\, \phi_* \subseteq \ker \psi_*$. We now show that $\ker \psi_* \subseteq \mathrm{im}\, \phi_*$. Let $f \in \ker \psi_* \subseteq \mathrm{Hom}_A(M, N)$. Then $\psi \circ f = \psi_* f = 0$, i.e.
$$\psi(f(m)) = 0 \quad \forall\, m \in M.$$
Hence $f(m) \in \ker \psi$ for every $m \in M$. Since $0 \longrightarrow N' \xrightarrow{\phi} N \xrightarrow{\psi} N''$ is exact, $\phi(N') = \ker \psi$, and the map $x \longmapsto \phi(x)$ gives an isomorphism
$$\bar{\phi} : N' \xrightarrow{\sim} \phi(N').$$
Since f takes values in $\ker \psi = \phi(N')$, if we define
$$g : M \longrightarrow N'$$
by the formula
$$g(x) = \bar{\phi}^{-1}(f(x)), \quad x \in M,$$
then $\phi \circ g = f$, i.e. $\phi_*(g) = f$. Thus $f \in \mathrm{im}\, \phi_*$.

This proves (a).

Remark: The above argument can be re-packaged invoking the universal properties of kernels, namely, if $K = \ker(\psi)$ and $f : M \longrightarrow N$ is an A-map s.t. $\psi \circ f = 0$, then $\exists !$ map $g : M \longrightarrow K$ s.t. $f = i \circ g$, where $i : K \longrightarrow N$ is the inclusion map $K \subseteq N$. Since
$$\begin{tikzcd} K \arrow[r, "i"] & N \\ N' \arrow[u, "\wr"] \arrow[r, "\phi"] & N \arrow[u, equal] \end{tikzcd}$$
commutes, we are done.

(b) The argument is dual to the one given in (a). Since $\psi : N \twoheadrightarrow N''$ is surjective, if $f \circ \psi = 0$ for an A-map $f : N'' \to M$ then $f = 0$. This means ψ^* is injective. For the remaining part, use the universal property of cokernels, namely if $C = \operatorname{coker}(\phi)$, $\pi : N \twoheadrightarrow C = N/(\operatorname{im}(\phi))$ the natural surjective map, and $g : N \to M$ an A-map such that $g \circ \phi = 0$, then $\exists!$ map $h : C \to M$ such that $g = h \circ \pi$.

$$\begin{tikzcd} N' \arrow[r, hook, "\phi"] \arrow[dr] & N \arrow[r, two heads, "\pi"] \arrow[d, "g"] & C \arrow[r, hook] \arrow[dl, dashed, "\exists! h"] & N'' \\ & M & & \end{tikzcd}$$

$g \circ \phi = 0$

The details are left to you. The diagram on the left may help.

The right exactness of $- \otimes_A N$

We need the following lemma

Lemma: Suppose we have a pair of A-maps

$$M \xrightarrow{\phi} N \xrightarrow{\psi} T$$

Then the sequence is exact (i.e. $\operatorname{im} \phi = \ker \psi$) if

$$\operatorname{Hom}_A(T, P) \xrightarrow{\psi^*} \operatorname{Hom}_A(N, P) \xrightarrow{\phi^*} \operatorname{Hom}_A(M, P)$$

is an exact sequence of A-module for every $P \in \operatorname{Mod}_A$.

Proof:

Take $P = T$. Then $\psi = \psi^*(1_P) \in \operatorname{im} \psi^* = \ker \phi^*$, i.e. $\psi^*(\phi) = 0$, i.e. $\psi \circ \phi = 0$. This means $\operatorname{im} \phi \subseteq \ker \psi$.

Next take $P = \operatorname{coker} \phi$. Let $\pi \in \operatorname{Hom}_A(N, P)$ be the natural map $N \xrightarrow{\pi} \operatorname{coker} \phi = P$. Now $\phi^*(\pi) = \pi \circ \phi = 0$. Thus $\pi \in \ker(\phi^*) = \operatorname{im}(\psi^*)$. This means $\exists\, h : T \to \operatorname{coker} \phi = P$ such that $h \circ \psi = \pi$. Thus if $n \in \ker \psi \subseteq N$, then $\pi(n) = (h \circ \psi)(n) = h(\psi(n)) = 0$. It follows that $n \in \operatorname{im} \phi$. Thus $\ker \psi \subseteq \operatorname{im} \phi$. //

We are now in a position to prove the right exactness of $- \otimes_A N$ (see next page).

Theorem: Let

$$N' \xrightarrow{\phi} N \xrightarrow{\psi} N'' \longrightarrow 0$$

be an exact sequence of A-modules. Then for any $M \in \mathrm{Mod}_A$ the induced sequence

$$M \otimes_A N' \xrightarrow{M \otimes_A \phi} M \otimes_A N \xrightarrow{M \otimes_A N} M \otimes_A N'' \longrightarrow 0$$

is exact.

Proof: Let $P \in \mathrm{Mod}_A$. Then

$$0 \to \mathrm{Hom}_A(N'', P) \xrightarrow{\psi^*} \mathrm{Hom}_A(N, P) \xrightarrow{\phi^*} \mathrm{Hom}_A(N', P)$$

is exact. This induces an exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(M, \mathrm{Hom}_A(N'', P)) \xrightarrow[\text{via } \psi]{} \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P)) \xrightarrow[\text{via } \phi]{} \mathrm{Hom}_A(M, \mathrm{Hom}_A(N', P))$$

By the Hom-$\otimes$ adjointness, this gives an exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(M \otimes_A N'', P) \xrightarrow[(M \otimes_A \psi)^*]{} \mathrm{Hom}_A(M \otimes_A N, P) \xrightarrow[(M \otimes_A \phi)^*]{} \mathrm{Hom}_A(M \otimes_A N', P).$$

The theorem follows from the previous lemma. //