HW 5 - SOLUTIONS

Throughout this assignment, A is a ring. If $A \subset B$ is a extension of rings and $x \in B$, then

$$A[x] := \{a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \mid a_i \in A, i = 1, \ldots, n\}.$$

$A[x]$ the smallest subring of B containing B and x.

Integral extensions. In the following exercises $A \subset B$ ia an extension of rings. If $\mathfrak{a}$ is an ideal of A, then anelement x of B is said to be *integral over* $\mathfrak{a}$ if there exists a positive integer n and elements $a_i \in \mathfrak{a}$, $i = 1, \ldots, n$, such that

$$x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0.$$

1. Show that if $x \in B$ is integral over A, then $A[x]$ is finitely generated as an A-module.

Solution: We know that $x^n + a_1x^{n-1} + \cdots + a_n = 0$ for some $a_1, \ldots, a_n \in A$, and some $n \geq 1$. Let M be the A-submodule of $A[x]$ generated by $1, x, \ldots, x^{n-1}$. It is clear that $x^{n+k} \in M$ for $k \geq 0$. In greater detail, since $x^n = -(a_1x^{n-1} + \cdots + a_{n-1}x + a_n)$, the statement is true for $k = 0$. For the same reason, $x(x^j) \in M$ for $j = 0, \ldots, n-1$, whence $xm \in M$ for every $m \in M$. It follows that $x^d m \in M$ for all $d \geq 0$ and $m \in M$. Taking $m = 1$, we see that $x^k \in M$ for all $k \geq 0$. Thus $A[x] \subset M$. On the other hand, clearly $M \subset A[x]$.

2. Show that if B is finitely generated over A as an A-module, then every element of B is integral over A. [**Hint:** Use the "determinant trick" introduced when proving Nakayama's Lemma (see pp. 4–6 of Lecture 5).]

Solution: Suppose $b_1, \ldots, b_d$ generate B as an A-module. We have a_{ij}, $1 \leq i, j \leq d$ such that $xb_i = \sum_j a_{ij}b_j$. The determinant trick shows that $\det(x\delta_{ij} - a_{ij}) = 0$. This gives the required integral equation.

3. Let $\bar{A}$ be the subset of B consisting of elements of B integral over A. Show that $\bar{A}$ is a subring of B containing A. (The ring $\bar{A}$ is called the *integral closure of A in B*.)

Solution: It is worth noting that $A \subset \bar{A}$. Let $x, y \in \bar{A}$. Then $A[x]$ is a finitely generated module over A by Problem 1, and since y is integral over $A[x]$ (being integral over A), the same reasoning shows that $A[x, y]$ is a finitely generated $A[x]$-module. It follows that $A[x, y]$ is a finitely generated A-module. By Problem 2, all elements of $A[x, y]$ are integral over A. In particular, $x \pm y$, and xy lie in $\bar{A}$. Thus $\bar{A}$ is closed under ring operations, whence it is a subring of B.

4. Let $x \in B$ and let $\mathfrak{a}$ be an ideal of A.

(a) Show that if x is integral over $\mathfrak{a}$, then $x \in \sqrt{\mathfrak{a}B}$,

(b) Suppose B is finitely generated over A as an A-module. If $x \in \sqrt{\mathfrak{a}B}$ then show that x is integral over $\mathfrak{a}$. [**Hint:** Use the "determinant trick".]

Solution:

(a) We have $x^n + a_1x^{n-1} + \cdots + a_n = 0$ for some $n \geq 1$ and some $a_i \in \mathfrak{a}$, $i = 1, \ldots, n$. Since $x^n = -(a_1x^{n-1} + \cdots + a_{n-1}x + a_n) \in \mathfrak{a}B$, it is clear that $x \in \sqrt{\mathfrak{a}B}$.

(b) Let B be generated as an A-module by $b_1, \ldots, b_d$. Now for some $n \geq 1$, $x^n \in \mathfrak{a}B$, say $x^n = \alpha_s\beta_s$, with $\alpha_s]in\mathfrak{a}$ and $\beta_s \in B$. Writing each of the β_s as linear combinations of the b_i, we see that $x^n = \sum_{r=1}^{d} y_ib_i$ with $y_i \in \mathfrak{a}$ for $1 \leq i \leq d$. Then $x^nb_i = \sum_{j-1}^{d} a_{ij}b_j$, for some $a_{ij} \in \mathfrak{a}$, $1 \leq i, j \leq d$. It follows that $\det(x^n\delta_{ij} - a_{ij}) = 0$, which means x is integral over $\mathfrak{a}$.

Homological Algebra. Let $\mathbf{C}(A)$ denote the category of (cochain) complexes of A-modules. Let $C^\bullet$ and $D^\bullet$ be complexes of A-modules. We write $\mathrm{Hom}_{\mathbf{C}(A)}(C^\bullet,\, D^\bullet)$, or simply $\mathrm{Hom}(C^\bullet, D^\bullet)$ if the context is clear, for the A-module of cochain maps from $C^\bullet$ to $D^\bullet$.[1]

In addition to $\mathrm{Hom}(C^\bullet, D^\bullet)$ we have another "Hom"–the so called *internal Hom*– between $C^\bullet$ and $D^\bullet$, namely the complex of A-modules $\mathrm{Hom}^\bullet_A(C^\bullet,\, D^\bullet)$ defined as follows: In degree n, $\mathrm{Hom}^\bullet_A(C^\bullet, D^\bullet)$ is given by

$$\mathrm{Hom}^n_A(C^\bullet, D^\bullet) = \prod_{j\in\mathbf{Z}} \mathrm{Hom}_A(C^j, D^{j+n})$$

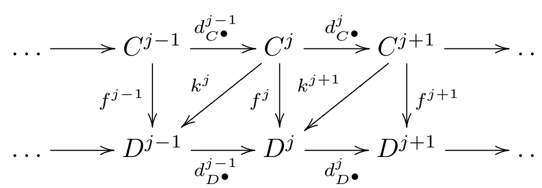
and the differential $d^n \colon \mathrm{Hom}^n_A(C^\bullet, D^\bullet) \to \mathrm{Hom}^{n+1}_A(C^\bullet, D^\bullet)$ takes $f = (f^j)_{j\in\mathbf{Z}}$ with $f^j \in \mathrm{Hom}_A(C^j, D^{j+n})$ to

$$d^n(f) = (d^{n+j}_{D^\bullet} \circ f^j + (-1)^{n+1} f^{j+1} \circ d^j_{C^\bullet})_{j\in\mathbf{Z}}.$$

We often write $\mathrm{Hom}^\bullet(C^\bullet, D^\bullet)$ for $\mathrm{Hom}^\bullet_A(C^\bullet, D^\bullet)$ if no confusion is likely to arise.

A map of complexes $f \colon C^\bullet \to D^\bullet$ is said to be *homotopic to zero*, written $f \sim 0$, if there exist A-maps $k^j \colon C^j \to D^{j-1}$, one for each $j \in \mathbf{Z}$, such that

$$d^{j-1}_{D^\bullet} \circ k^j + k^{j+1} \circ d^j_{C^\bullet} = f^j.$$



If f is homotopic to zero, clearly so is $-f$. Let f, g be two elements of $\mathrm{Hom}(C^\bullet, D^\bullet)$. We say f *is homotopic to* g if $f - g \sim 0$. In this case we write $f \sim g$.

Recall that for $C^\bullet \in \mathbf{C}(A)$, $Z^n(C^\bullet)$ is the A submodule of C^n consisting of n-cocycles, and $B^n(C^\bullet)$, the submodule of n-coboundaries.

Fix $C^\bullet, D^\bullet \in \mathbf{C}(A)$ in what follows. In what follows, keep in mind the difference between $\mathrm{Hom}(C^\bullet, D^\bullet)$ and $\mathrm{Hom}^\bullet(C^\bullet, D^\bullet)$.

5. Show that $\mathrm{Hom}^\bullet(C^\bullet, D^\bullet)$ is a complex.

[1]We use the terms "cochain map" and "map of complexes" interchangeably.

Solution: Let $f = (f^j)_{j\in\mathbf{Z}} \in \mathrm{Hom}^n_A(C^\bullet,\, D^\bullet)$ and $g = d^n f$, and write g^j for the j^{th} component of g, i.e. $g^j : C^j \to D^{j+n+1}$ and $g = (g^j)_{j\in\mathbf{Z}}$. Then

$$d^{n+1}g = (d_D^{n+1+j}g^j + (-1)^{n+2}g^{j+1}d_C^j)_{j\in\mathbf{Z}}.$$

The computations can be broken up into two parts:

$$\begin{aligned} d_D^{n+1+j}g^j &= d_D^{n+j+1}(d_D^{n+j+1}f^j + (-1)^{n+1}f^{j+1}d_C^j \\ &= (1)^{n+1}d_D^{n+j+1}f^{j+1}d_C^j \end{aligned}$$

and

$$\begin{aligned} (-1)^{n+2}g^{j+1}d_C^j &= (-1)^{n+2}(d_D^{n+j+1}f^{j+1} + (-1)^{n+1}f^{j+2}d_C^{j+1})d_C^j \\ &= (1)^{n+2}d_D^{n+j+1}f^{j+1}d_C^j \end{aligned}$$

On adding, we see that $d_D^{n+1+j}g^j + (-1)^{n+2}g^{j+1}d_C^j = 0$ for $j \in \mathbf{Z}$, i.e. $d^{n+1}d^n f = 0$ for every $f \in \mathrm{Hom}^n_A(C^\bullet,\, D^\bullet)$ and every $n \in \mathbf{Z}$. Thus $\mathrm{Hom}^\bullet_A(C^\bullet,\, D^\bullet)$ is a complex.

6. Show that $Z^0(\mathrm{Hom}^\bullet(C^\bullet, D^\bullet))$ is the set of cochain maps from $C^\bullet$ to $D^\bullet$. In other words, show $Z^0(\mathrm{Hom}^\bullet(C^\bullet, D^\bullet)) = \mathrm{Hom}(C^\bullet, D^\bullet)$.

Solution: If $f \in \mathrm{Hom}^0_A(C^\bullet,\, D^\bullet)$, say $f = (f^j)_{j\in\mathbf{Z}}$, then $d^0 f = (d_D^j f^j - f^{j+1}d_C^j)_{j\in\mathbf{Z}}$. It follows that f is a 0-cocycle in $\mathrm{Hom}^\bullet_A(C^\bullet,\, D^\bullet)$ if and only if $d_D^j f^j - f^{j+1}d_C^j = 0$ for all $j \in \mathbf{Z}$, i.e. if and only if f is a cochain map.

7. Show that $B^0(\mathrm{Hom}^\bullet(C^\bullet, D^\bullet))$ is the subset of $\mathrm{Hom}(C^\bullet, D^\bullet)$ consisting of cochain maps which are homotopic to zero.

Solution: An element $f = (f^j)_{j\in\mathbf{Z}} \in \mathrm{Hom}^0_A(C^\bullet,\, D^\bullet)$ is a 0-coboundary if and only if $f = d^{-1}k$, for some $k \in \mathrm{Hom}^{-1}_A(C^\bullet,\, D^\bullet)$. Write $k = (k^j)_{j\in\mathbf{Z}}$. The equation $f = d^{-k}f$ translates to $(f^j)_{j\in\mathbf{Z}} = (d_D^{j-1}k^j + k^{j+1}d_C^j)_{j\in\mathbf{Z}}$ which amounts to saying $f \sim 0$.

8. Suppose $f, g \in \mathrm{Hom}(C^\bullet, D^\bullet)$, with $f \sim g$. Show that $\mathrm{H}^n(f) = \mathrm{H}^n(g)$ for all $n \in \mathbf{Z}$.

Solution: There exists $k = (k^j)_{j\in\mathbf{Z}} \in \mathrm{Hom}^{-1}_A(C^\bullet,\, D^\bullet)$ such that

$$f^n - g^n = d_D^{n-1}k^n + k^{n+1}d_C^n, \qquad n \in \mathbf{Z}.$$

Fix $n \in \mathbf{Z}$. Since $d_C^n(Z^n(C^\bullet)) = 0$ (by definition of $(Z^n(C^\bullet))$), it follows that $(f^n - g^n)|_{Z^n(C^\bullet)} = (d_D^{n-1}k^n)|_{Z^n(C^\bullet)}$. Thus $\mathrm{Im}(f^n - g^n)|_{Z^n(C^\bullet)} \subset B^n(D^\bullet)$. It is immediate that $\mathrm{H}^n(f - g) = 0$.