HW 5

Due date: March 21, 2022 (either online or in class)

Throughout this assignment, A is a ring. If $A \subset B$ is a extension of rings and $x \in B$, then

$$A[x] := \{a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \mid a_i \in A, i = 1, \ldots, n\}.$$

$A[x]$ the smallest subring of B containing B and x.

Integral extensions. In the following exercises $A \subset B$ ia an extension of rings. If $\mathfrak{a}$ is an ideal of A, then anelement x of B is said to be *integral over* $\mathfrak{a}$ if there exists a positive integer n and elements $a_i \in \mathfrak{a}$, $i = 1, \ldots, n$, such that

$$x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0.$$

1. Show that if $x \in B$ is integral over A, then $A[x]$ is finitely generated as an A-module.

2. Show that if B is finitely generated over A as an A-module, then every element of B is integral over A. [**Hint:** Use the "determinant trick" introduced when proving Nakayama's Lemma (see pp. 4–6 of Lecture 5).]

3. Let $\bar{A}$ be the subset of B consisting of elements of B integral over A. Show that $\bar{A}$ is a subring of B containing A. (The ring $\bar{A}$ is called the *integral closure of A in B*.)

4. Let $x \in B$ and let $\mathfrak{a}$ be an ideal of A.
- **(a)** Show that if x is integral over $\mathfrak{a}$, then $x \in \sqrt{\mathfrak{a}B}$,
- **(b)** Suppose B is finitely generated over A as an A-module. If $x \in \sqrt{\mathfrak{a}B}$ then show that x is integral over $\mathfrak{a}$. [**Hint:** Use the "determinant trick".]

Homological Algebra. Let $\mathbf{C}(A)$ denote the category of (cochain) complexes of A-modules. Let $C^\bullet$ and $D^\bullet$ be complexes of A-modules. We write $\mathrm{Hom}_{\mathbf{C}(A)}(C^\bullet, D^\bullet)$, or simply $\mathrm{Hom}(C^\bullet, D^\bullet)$ if the context is clear, for the A-module of cochain maps from $C^\bullet$ to $D^\bullet$.[1]

In addition to $\mathrm{Hom}(C^\bullet, D^\bullet)$ we have another "Hom"–the so called *internal Hom*– between $C^\bullet$ and $D^\bullet$, namely the complex of A-modules $\mathrm{Hom}^\bullet_A(C^\bullet, D^\bullet)$ defined as follows: In degree n, $\mathrm{Hom}^\bullet_A(C^\bullet, D^\bullet)$ is given by

$$\mathrm{Hom}^n_A(C^\bullet, D^\bullet) = \prod_{j\in\mathbf{Z}} \mathrm{Hom}_A(C^j, D^{j+n})$$

and the differential $d^n \colon \mathrm{Hom}^n_A(C^\bullet, D^\bullet) \to \mathrm{Hom}^{n+1}_A(C^\bullet, D^\bullet)$ takes $f = (f^j)_{j\in\mathbf{Z}}$ with $f^j \in \mathrm{Hom}_A(C^j, D^{j+n})$ to

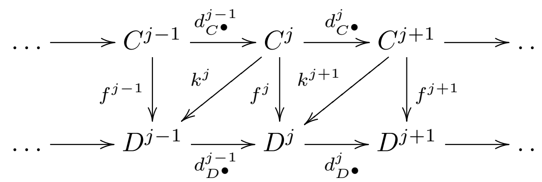
$$d^n(f) = (d^{n+j}_{D^\bullet} \circ f^j + (-1)^{n+1} f^{j+1} \circ d^j_{C^\bullet})_{j\in\mathbf{Z}}.$$

[1] We use the terms "cochain map" and "map of complexes" interchangeably.

We often write $\mathrm{Hom}^\bullet(C^\bullet, D^\bullet)$ for $\mathrm{Hom}_A^\bullet(C^\bullet, D^\bullet)$ if no confusion is likely to arise.

A map of complexes $f\colon C^\bullet \to D^\bullet$ is said to be *homotopic to zero*, written $f \sim 0$, if there exist A-maps $k^j : C^j \to D^{j-1}$, one for each $j \in \mathbf{Z}$, such that

$$d_{D^\bullet}^{j-1} \circ k^j + k^{j+1} \circ d_{C^\bullet}^j = f^j.$$



If f is homotopic to zero, clearly so is $-f$. Let f, g be two elements of $\mathrm{Hom}(C^\bullet, D^\bullet)$. We say f *is homotopic to* g if $f - g \sim 0$. In this case we write $f \sim g$.

Recall that for $C^\bullet \in \mathbf{C}(A)$, $Z^n(C^\bullet)$ is the A submodule of C^n consisting of n-cocycles, and $B^n(C^\bullet)$, the submodule of n-coboundaries.

Fix $C^\bullet, D^\bullet \in \mathbf{C}(A)$ in what follows. In what follows, keep in mind the difference between $\mathrm{Hom}(C^\bullet, D^\bullet)$ and $\mathrm{Hom}^\bullet(C^\bullet, D^\bullet)$.

5. Show that $\mathrm{Hom}^\bullet(C^\bullet, D^\bullet)$ is a complex.

6. Show that $Z^0(\mathrm{Hom}^\bullet(C^\bullet, D^\bullet))$ is the set of cochain maps from $C^\bullet$ to $D^\bullet$. In other words, show $Z^0(\mathrm{Hom}^\bullet(C^\bullet, D^\bullet)) = \mathrm{Hom}(C^\bullet, D^\bullet)$.

7. Show that $B^0(\mathrm{Hom}^\bullet(C^\bullet, D^\bullet))$ is the subset of $\mathrm{Hom}(C^\bullet, D^\bullet)$ consisting of cochain maps which are homotopic to zero.

8. Suppose $f, g \in \mathrm{Hom}(C^\bullet, D^\bullet)$, with $f \sim g$. Show that $\mathrm{H}^n(f) = \mathrm{H}^n(g)$ for all $n \in \mathbf{Z}$.