HW 4 - SOLUTIONS

Throughout this assignment, A is a ring, and M an A-module. As always, a *proper* submodule of M is a submodule which is not equal to M.

Annihilators and the support of a module. The *annihilator* $\mathrm{ann}(m)$ of an element $m \in M$ is the set of elements $a \in A$ such that $am = 0$. It is clear that $\mathrm{ann}(m)$ is an ideal of A. If we wish to specify the ring in which the annihilator is computed (M could be regarded as a module over any ring which maps to A), then we write $\mathrm{ann}_A(M)$ for $\mathrm{ann}(M)$. The *annhilator of M in A* is the ideal $\mathrm{ann}(M)$, or more accurately, $\mathrm{ann}_A(M)$, defined by the formula

$$\mathrm{ann}(M) = \mathrm{ann}_A(M) := \bigcap_{m \in M} \mathrm{ann}(m).$$

If $\{m_\lambda \mid \lambda \in \Lambda\}$ is a set of generators of M, it is clear that $\mathrm{ann}(M) = \bigcap_{\lambda\in\Lambda} \mathrm{ann}(m_\lambda)$.

The *support of M over A*, or simply the *support of M* if the context is clear, denoted $\mathrm{Supp}(M)$, is:

$$\mathrm{Supp}(M) := \{\mathfrak{p} \in \mathrm{Spec}\, A \mid M_\mathfrak{p} \neq 0\}.$$

Once again, if we wish to specify the ring A, we write $\mathrm{Supp}_A(M)$ for the support of M over A.

1. If $\mathrm{Supp}(M) = \emptyset$ then show that $M = 0$. [**Hint:** Reduce to the case where M is finitely generated. Next show that there exist $f_\lambda \in A$, λ varying in some index set Λ, such that $\{D(f_\lambda) \mid \lambda \in \Lambda\}$ is an open cover of $X = \mathrm{Spec}\, A$, and $M_{f_\lambda} = 0$ for every λ. Use the quasi-compactness of X to find elements $g_1, \ldots, g_d \in A$, such that $g_i \in \mathrm{Ann}(M)$ and $\cup_i D(g_i) = X$.]

Solution: The hint was unnecessary as I now realise. For $\mathfrak{p} \in \mathrm{Spec}(A)$, let $1_\mathfrak{p}$ be the multiplicative identity in $A_\mathfrak{p}$. Let $m \in M$. Then $m/1_\mathfrak{p} = 0 \in A_\mathfrak{p}$ for every $\mathfrak{p} \in \mathrm{Spec}(A)$. For each prime ideal $\mathfrak{p}$ there then exists an elements $s_\mathfrak{p} \in A \smallsetminus \mathfrak{p}$ such that $s_\mathfrak{p} m = 0$. Since $\mathfrak{p} \in A_{s_\mathfrak{p}}$, it is clear that $\{D(s_\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}(A)\}$ is an open cover of $\mathrm{Spec}(A)$. We can find a finite subcover of this. This means we can find $f_1, \ldots, f_d \in A$ such that $f_i m = 0$ and $\cup_{i=1}^d D(f_i) = \mathrm{Spec}(A)$. Since $\langle f_1, \ldots, f_d\rangle = A$ we have $a_1, \ldots, a_d \in A$ such that $\sum_{i=1}^d a_i f_i = 1$. It follows that $m = 1 \cdot m = \sum_{i=1}^d a_i f_i m = 0$. $\square$

2. Suppose M is finitely generated.

(a) Show that $V(\mathrm{ann}(M)) = \mathrm{Supp}(M)$.

(b) Show that

$$\sqrt{\mathrm{ann}(M)} = \bigcap_{\mathfrak{p}\in\mathrm{Supp}(M)} \mathfrak{p}.$$

Solution: Let $M = \langle m_1, \ldots, m_d\rangle$. Let $\mathfrak{p} \in \mathrm{Spec}(A)$. Let $S = A \smallsetminus \mathfrak{p}$. It is evident that $M_\mathfrak{p} = 0$ if and only if $m_i/1 = 0$ in $A_\mathfrak{p}$. This happens if and only of there exist $s_i \in S$ such that $s_i m_i = 0$ for $i = 1, \ldots, d$. The last condition is equivalent to the condition that there exists $s \in S$ such that $sm_i = 0$. Indeed, if such an $s \in S$ exists, then we can choose s_i to equal s for all $i = 1, \ldots, d$, and conversely, if there exist $s_i \in S$ such that $s_i m_i = 0$, $i = 1, \ldots, d$, then we can set $s = s_1 \ldots s_d$. Now $sm_i = 0$ for all i if and only if $s \in \operatorname{ann}(M)$. Thus the condition $M_\mathfrak{p} = 0$ is equivalent to saying there exists $s \in S \cap \operatorname{ann}(M)$, i.e. $\operatorname{ann}(M) \not\subset \mathfrak{p}$. It follows that $M_\mathfrak{p} \neq 0$ if and only if $\operatorname{ann}(M) \subset \mathfrak{p}$. This proves (a). Part (b) is a direct consequence, since $\sqrt{\operatorname{ann}(M)} = \bigcap_{\mathfrak{p} \in V(\operatorname{ann}(M))} \mathfrak{p}$.

There is a second approach which is more transparent. Suppose $M_\mathfrak{p} \neq 0$. Then there is some $m \in M \smallsetminus \{0\}$ such that $m/1 \neq 0$. Clearly $sm \neq 0$ for any $s \in S$. Thus $S \cap \operatorname{ann}(m) = \emptyset$, i.e. $\operatorname{ann}(m) \subset \mathfrak{p}$, which means $\operatorname{ann}(M) \subset \mathfrak{p}$. This inclusion does not require finite generation of M. However, to show that if $\mathfrak{p} \supset \operatorname{ann}(M)$, then $M_\mathfrak{p} \neq 0$, we do require finite generation. Clearly $\operatorname{ann}(M) = \cap_{1=1}^{d} \operatorname{ann}(m_i)$. By the very first proposition of Lecture 8 we have $\operatorname{ann}(m_i) \subset \mathfrak{p}$ for some $i \in \{1, \ldots, d\}$. Thus if $s \in S$, then $s \notin \operatorname{ann}(m_i)$, whence $m_i/1 \neq 0$. ☐

Irreducible submodules. A submodule N of M is called *irreducible in M* (or simply *irreducible* if the context is clear) if is satisfies the following condition: If there exist two submodules N_1 and N_2 of M such that $N = N_1 \cap N_2$, then $N = N_1$ or $N = N_2$.

3. Let A be Noetherian and M finitely generated. Show that every proper submodule of M can be written as a finite intersection of irreducible modules. [**Hint:** The proof of Proposition 2.1.3 of Lecture 12 may help.]

Solution: Let Σ be the collection of proper submodules of M which cannot be written as a finite intersection of irreducible modules. Suppose Σ is nonempty. Then there exists a maximal element N of Σ. N cannot be irreducible, and so $N = N_1 \cap N_2$, where neither N_1 nor N_2 is N. This means N_i are proper submodules of M, and by maximality of N, each of them is a finite intersection of irreducible submodules, whence N is a finite intersection of irreducible submodules. This contradicts the fact that N is a member of Σ. Thus Σ is empty. ☐

Associated primes. A prime ideal $\mathfrak{p}$ of A is said to be *associated to M* if there exists $m \in M$ such that $\mathfrak{p} = \operatorname{ann}(m)$. We denote by $\operatorname{Ass}_A(M)$ the collection of primes associated to M.[1] If the context is clear, we write $\operatorname{Ass}(M)$ for $\operatorname{Ass}_A(M)$.

4. **(a)** Let $M = A/\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal of A. Show that for every element $m \in M \smallsetminus \{0\}$, $\operatorname{ann}(m) = \mathfrak{p}$. Conclude that $\operatorname{Ass}_A(A/\mathfrak{p}) = \{\mathfrak{p}\}$.
(b) Calculate $\operatorname{Ass}_{A/\mathfrak{p}}(A/\mathfrak{p})$.
(c) Prove that $\mathfrak{p} \in \operatorname{Ass}(M)$ if and only if there is an injective A-module map from $A/\mathfrak{p}$ into M.

Solution:

(a) Suppose m is non-zero in M. Then $m = a + \mathfrak{p}$, with $a \notin \mathfrak{p}$. It is clear that $xm = 0$ for $x \in A$ if and only if $xa \in \mathfrak{p}$, and since $a \notin \mathfrak{p}$, the last condition is equivalent to saying $x \in \mathfrak{p}$. Thus $\operatorname{ann}(m) = \mathfrak{p}$. From this it is clear that $\operatorname{Ass}(M) = \{\mathfrak{p}\}$. ☐

(b) Since $A/\mathfrak{p}$ is an integral domain, $\operatorname{ann}_{A/\mathfrak{p}}(m) = 0$ for any non-zero element of $A/\mathfrak{p}$. Since 0 is a prime ideal of $A/\mathfrak{p}$, it follows that $\operatorname{Ass}_{A/\mathfrak{p}}(A/\mathfrak{p}) = \{\langle 0 \rangle\}$. ☐

[1]Instead of "$\mathfrak{p}$ is a prime associated to M" we often say "$\mathfrak{p}$ is an associated prime of M".

(c) Suppose we have an injective map $A/\mathfrak{p} \hookrightarrow M$. Then for any nonzero element m in the image of $A/\mathfrak{p}$, we have $\operatorname{ann}(m) = \mathfrak{p}$ by part (a). Thus $\mathfrak{p} \in \operatorname{Ass}(M)$. Conversely, if $\mathfrak{p} \in \operatorname{Ass}(M)$, there exists a nonzero $m \in M$ such that $\mathfrak{p} = \operatorname{ann}(m)$. Let $f\colon A \to M$ be the A-map given by $f(a) = am$. Then $\ker(f) = \operatorname{ann}(m) = \mathfrak{p}$, and hence we have an injective map $A/\mathfrak{p} \hookrightarrow M$. □

5. Let

$$0 \longrightarrow N \longrightarrow M \longrightarrow T \longrightarrow 0$$

be an exact sequence of A-modules.

(a) Show that $\operatorname{Ass}(N) \subset \operatorname{Ass}(M)$.

(b) Show that $\operatorname{Ass}(M) \subset \operatorname{Ass}(N) \cup \operatorname{Ass}(T)$. [**Hint:** If $\mathfrak{p} \in \operatorname{Ass}(M) \smallsetminus \operatorname{Ass}(N)$, then show that $\mathfrak{p} \in \operatorname{Ass}(T)$.]

Solution:

(a) Since $N \to M$ is injective, we may regard N as a submodule of M. The annihilator of any element of N is the same as the annihilator of its image in M, and hence $\operatorname{Ass}(N) \subset \operatorname{Ass}(M)$. □

(b) Suppose $\mathfrak{p} \in \operatorname{Ass}(M) \smallsetminus \operatorname{Ass}(N)$. Let $m \in M$ be such that $\mathfrak{p} = \operatorname{ann}(m)$. Since $\mathfrak{p} \notin \operatorname{Ass}(N)$, m does not lie in N, and so the image t of m in T is non-zero. Moreover $\mathfrak{p} = \operatorname{ann}(m) \subset \operatorname{ann}(t)$. Let $x \in \operatorname{ann}(t)$. Then $xm \in N$ and $\operatorname{ann}(xm) \supset \operatorname{ann}(m) = \mathfrak{p}$. This is a strict inclusion since $\mathfrak{p}$ is not an associated prime of N. Therefore there exists an element $s \in \operatorname{ann}(xm) \smallsetminus \mathfrak{p}$. Since $sx \in \operatorname{ann}(m) = \mathfrak{p}$ and $s \notin \mathfrak{p}$, it follows that $x \in \mathfrak{p}$. Thus $\mathfrak{p} = \operatorname{ann}(t)$. □

6. Prove that if A is Noetherian and $M \neq 0$ then $\operatorname{Ass}(M) \neq \emptyset$. [**Hint:** Apply the maximality condition to the set of ideals which are annihilators of non-zero elements.]

Solution: Following the hint, suppose $\mathfrak{a}$ is a maximal member of the set if annihilators of non-zero elements of M. Say $\mathfrak{a} = \operatorname{ann}(m)$. If $a \notin \mathfrak{a}$ then $\operatorname{ann}(am) = \mathfrak{a}$, since $\operatorname{ann}(am) \supset \operatorname{ann}(m) = \mathfrak{a}$, and $\mathfrak{a}$ is maximal amongst annihilators of nonzero elements of M. Accordingly, if $ab \in \mathfrak{a}$ and $a \notin \mathfrak{a}$, then $b \in \operatorname{ann}(am) = \mathfrak{a}$. Thus $\mathfrak{a}$ is prime, and so $\mathfrak{a} \in \operatorname{Ass}(M)$.

7. A *zero divisor* of M is an element $a \in A$ such that $am = 0$ for some non-zero element m of M. Let $\operatorname{ZD}(M)$ denote the set of zero divisors of M. If A is Noetherian, show that

$$\operatorname{ZD}(M) = \bigcup_{\mathfrak{p} \in \operatorname{Ass}(M)} \mathfrak{p}.$$

Solution: It is clear that $\operatorname{ZD}(M) = \bigcup_{m \neq 0} \operatorname{ann}(m)$. In particular we have the inclusion $\bigcup_{\mathfrak{p} \in \operatorname{Ass}(M)} \mathfrak{p} \subset \operatorname{ZD}(M)$. (For this inclusion, the Noetherian-ness of A plays no role.)

Since A is Noetherian, if $\Sigma = \{\operatorname{ann}(m) \mid m \neq 0\}$, and $\Sigma_{\max}$ is the subset of Σ consisting of maximal elements of Σ, then every member of Σ is contained in a member of $\Sigma_{\max}$. Thus $\operatorname{ZD}(M) = \bigcup_{\mathfrak{p} \in \Sigma_{\max}} \mathfrak{p}$. From the solution to the previous problem we see that $\Sigma_{\max} \subset \operatorname{Ass}(M)$, whence $\operatorname{ZD}(M) \subset \bigcup_{\mathfrak{p} \in \operatorname{Ass}(M)} \mathfrak{p}$. □

8. Let A be Noetherian and M finitely generated.

(a) Show that we have a descending chain of submodules

$$M = M_0 \supset M_1 \supset \cdots \supset M_n = 0$$

such that $M_i/M_{i+1} \cong A/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \operatorname{Spec} A$, $i = 0, \ldots, n-1$.

(b) Show that $\operatorname{Ass}(M)$ is a finite set. [**Hint:** Use part **(b)** of Problem **5**.]

Solution: We avoid annoying trivialities like M being zero. Note that if M is non zero, so is A.

(a) Equivalently, it is enough to show that there is an increasing sequence of A-modules $0 = N_0 \subset N_1 \subset \ldots N_d = M$ such that $N_/N_{i-1} \cong A/\mathfrak{q}_i$ for some $\mathfrak{q}_i \in \operatorname{Spec}(A)$, $i = 1, \ldots, d$. Pick $\mathfrak{q}_1$ in the nonempty set $\operatorname{Ass}(M)$. By part **(c)** of Problem **4**, we have an injective map $A/\mathfrak{q}_1 \hookrightarrow M$. Let N_1 be the image of $A/\mathfrak{q}_1$ in M. Suppose we have N_j, $1 \le j \le i$ such that $N_0 = 0$, $N_{j-1} \subset N_j$ and $N_j/N_{j-1}A/\mathfrak{q}_j$ for $j = 1, \ldots, i$. If $N_i = M$ we are done. If not, pick $\mathfrak{q}_{i+1} \in \operatorname{Ass}(M/N_i)$. We have a copy of $A/\mathfrak{q}_{i+1}$ in M/N_i, and this copy must be of the form N_{i+1}/N_i for a submodule N_{i+1} of M containing N_i. Since A is Noetherian and M is finitely generated, the ascending chain $N_0 \subset N_1 \subset \ldots N_i \subset \ldots$ must become stationary, and we are done. □

(b) Let $M_0, \ldots, M_n$ and $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be as in the statement of part (a). Applying part **(b)** of Problem **5** successively to the exact sequences $0 \to M_{i+1} \to M_i \to M_i/M_{i+1} \to 0$ we see that $\operatorname{Ass}(M) \subset \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. □