HW 4

Due date: March 3, 2022 (either online or in class)

Throughout this assignment, A is a ring, and M an A-module. As always, a *proper* submodule of M is a submodule which is not equal to M.

Annihilators and the support of a module. The *annihilator* $\operatorname{ann}(m)$ of an element $m \in M$ is the set of elements $a \in A$ such that $am = 0$. It is clear that $\operatorname{ann}(m)$ is an ideal of A. If we wish to specify the ring in which the annihilator is computed (M could be regarded as a module over any ring which maps to A), then we write $\operatorname{ann}_A(M)$ for $\operatorname{ann}(M)$. The *annhilator of M in A* is the ideal $\operatorname{ann}(M)$, or more accurately, $\operatorname{ann}_A(M)$, defined by the formula

$$\operatorname{ann}(M) = \operatorname{ann}_A(M) := \bigcap_{m \in M} \operatorname{ann}(m).$$

If $\{m_\lambda \mid \lambda \in \Lambda\}$ is a set of generators of M, it is clear that $\operatorname{ann}(M) = \bigcap_{\lambda \in \Lambda} \operatorname{ann}(m_\lambda)$.

The *support of M over A*, or simply the *support of M* if the context is clear, denoted $\operatorname{Supp}(M)$, is:

$$\operatorname{Supp}(M) := \{\mathfrak{p} \in \operatorname{Spec} A \mid M_{\mathfrak{p}} \neq 0\}.$$

Once again, if we wish to specify the ring A, we write $\operatorname{Supp}_A(M)$ for the support of M over A.

1. If $\operatorname{Supp}(M) = \emptyset$ then show that $M = 0$. [**Hint:** Reduce to the case where M is finitely generated. Next show that there exist $f_\lambda \in A$, λ varying in some index set Λ, such that $\{D(f_\lambda) \mid \lambda \in \Lambda\}$ is an open cover of $X = \operatorname{Spec} A$, and $M_{f_\lambda} = 0$ for every λ. Use the quasi-compactness of X to find elements $g_1, \ldots, g_d \in A$, such that $g_i \in \operatorname{Ann}(M)$ and $\cup_i D(g_i) = X$.]

2. Suppose M is finitely generated.
(a) Show that $V(\operatorname{ann}(M)) = \operatorname{Supp}(M)$.
(b) Show that

$$\sqrt{\operatorname{ann}(M)} = \bigcap_{\mathfrak{p} \in \operatorname{Supp}(M)} \mathfrak{p}.$$

Irreducible submodules. A submodule N of M is called *irreducible in M* (or simply *irreducible* if the context is clear) if is satisfies the following condition: If there exist two submodules N_1 and N_2 of M such that $N = N_1 \cap N_2$, then $N = N_1$ or $N = N_2$.

3. Let A be Noetherian and M finitely generated. Show that every proper submodule of M can be written as a finite intersection of irreducible modules. [**Hint:** The proof of Proposition 2.1.3 of Lecture 12 may help.]

Associated primes. A prime ideal $\mathfrak{p}$ of A is said to be *associated to* M if there exists $m \in M$ such that $\mathfrak{p} = \operatorname{ann}(m)$. We denote by $\operatorname{Ass}_A(M)$ the collection of primes associated to M.[1] If the context is clear, we write $\operatorname{Ass}(M)$ for $\operatorname{Ass}_A(M)$.

4. **(a)** Let $M = A/\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal of A. Show that for every element $m \in M \setminus \{0\}$, $\operatorname{ann}(m) = \mathfrak{p}$. Conclude that $\operatorname{Ass}_A(A/\mathfrak{p}) = \{\mathfrak{p}\}$.
(b) Calculate $\operatorname{Ass}_{A/\mathfrak{p}}(A/\mathfrak{p})$.
(c) Prove that $\mathfrak{p} \in \operatorname{Ass}(M)$ if and only if there is an injective A-module map from $A/\mathfrak{p}$ into M.

5. Let

$$0 \longrightarrow N \longrightarrow M \longrightarrow T \longrightarrow 0$$

be an exact sequence of A-modules.
(a) Show that $\operatorname{Ass}(N) \subset \operatorname{Ass}(M)$.
(b) Show that $\operatorname{Ass}(M) \subset \operatorname{Ass}(N) \cup \operatorname{Ass}(T)$. [**Hint:** If $\mathfrak{p} \in \operatorname{Ass}(M) \setminus \operatorname{Ass}(N)$, then show that $\mathfrak{p} \in \operatorname{Ass}(T)$.]

6. Prove that if A is Noetherian and $M \neq 0$ then $\operatorname{Ass}(M) \neq \emptyset$. [**Hint:** Apply the maximality condition to the set of ideals which are annihilators of non-zero elements.]

7. A *zero divisor* of M is an element $a \in A$ such that $am = 0$ for some non-zero element m of M. Let $\operatorname{ZD}(M)$ denote the set of zero divisors of M. If A is Noetherian, show that

$$\operatorname{ZD}(M) = \bigcup_{\mathfrak{p} \in \operatorname{Ass}(M)} \mathfrak{p}.$$

8. Let A be Noetherian and M finitely generated.
(a) Show that we have a descending chain of submodules

$$M = M_0 \supset M_1 \supset \cdots \supset M_n = 0$$

such that $M_i/M_{i+1} \cong A/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \operatorname{Spec} A$, $i = 0, \ldots, n-1$.
(b) Show that $\operatorname{Ass}(M)$ is a finite set. [**Hint:** Use part **(b)** of Problem **5**.]

[1] Instead of "$\mathfrak{p}$ is a prime associated to M" we often say "$\mathfrak{p}$ is an associated prime of M".