HW 3 - SOLUTIONS

Due date: Feb 17, 2022 (to be handed over in class)

Throughout, if A is a ring and $X = \operatorname{Spec} A$. The conventions used in lectures and the previous homework assignments remain in force. A reminder: if $f \in A$, then

$$D(f) := \{\mathfrak{p} \in X \mid f \notin \mathfrak{p}\}.$$

According to what was done (or will be done) in the lectures, $D(f)$ is homeomorphic to $\operatorname{Spec} A_f$ and in fact $D(f)$ is the homeomorphic image of $\operatorname{Spec} A_f$ under the natural map $\operatorname{Spec} A_f \to \operatorname{Spec} A = X$ induced by the localization map $A \to A_f$.

1. For a multiplicatively closed set S in A, we call the set

$$\overline{S} := \{a \in A \mid ab \in S \text{ for some } b \in A\}$$

the *saturation* of S. Let $f, g \in A$ and $M \in \mathrm{Mod}_A$. Fix a multiplicative system S.

- **(a)** Show that $\overline{S}$ is multiplicatively closed and that $\overline{S}^{-1}A = S^{-1}A$.
- **(b)** Let T be a multiplicative system such that $S \subset T \subset \overline{S}$. Show that $\overline{T} = \overline{S}$. In particular, conclude that $\overline{\overline{S}} = \overline{S}$.
- **(c)** Let T be another multiplicative system, and let ST be the multiplicative system $ST := \{st \in A \mid s \in S,\ t \in T\}$. Show that $(ST)^{-1}M = T^{-1}(S^{-1}M)$.
- **(d)** Show that if $d \geq 1$ then $M_{f^d} = M_f$.
- **(e)** Show that $(M_f)_g = M_{fg}$.

Solution: Most of the parts have easy solutions once we observe that $\overline{S}$ is the complement of the union of prime ideals which are disjoint from S. If $a \notin \overline{S}$, then by definition of $\overline{S}$, $\langle a \rangle \cap S = \emptyset$. Now Zornify. Let $\mathfrak{a}$ be a maximal ideal of A containing $\langle a \rangle$ and not meeting S. Standard arguments show that $\mathfrak{a}$ is prime. Conversely, if $a \in \mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal such that $\mathfrak{p} \cap S = \emptyset$, then $ab \in \mathfrak{p}$ for every $b \in A$, and hence $ab \notin S$ for any $b \in A$, which means $a \notin \overline{S}$. Thus if $\Sigma = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid \mathfrak{p} \cap S = \emptyset\}$, then

$$(*) \qquad \overline{S} = A \smallsetminus \bigcup_{\mathfrak{p} \in \Sigma} \mathfrak{p}.$$

- **(a)** From $(*)$ it is clear that $\overline{S}$ is a multiplicative system since it is the intersection of multiplicative systems. Next, we know (from Problem 6 of HW 1) that prime ideals of A which do not meet S are in bijective correspondence with prime ideals of $S^{-1}A$, the correspondence being $\mathfrak{p} \mapsto S^{-1}\mathfrak{p}$, with the inverse map being $\mathfrak{q} \mapsto i_A^{-1}(\mathfrak{q})$, where $i_A \colon A \to S^{-1}A$ is the localization map. It follows that if $t \in \overline{S}$ then $t/1 \in S^{-1}A$ is a unit in $S^{-1}A$ since, according to $(*)$, and our just given description of prime ideals in $S^{-1}A$, it does not lie in any prime ideal of $S^{-1}A$. It is immediate that $i_A \colon A \to S^{-1}A$ has the universal property of localization for $\overline{S}^{-1}A$.
- **(b)** Follows from $(*)$.

(c) Consider the composite of localization maps

$$A \xrightarrow{f_1} S^{-1}A \xrightarrow{f_2} T^{-1}(S^{-1}A).$$

Let $f = f_1 \circ f_1$. It is clear that $f(st)$ is a unit in $T^{-1}(S^{-1}A)$ for every $s \in S$ and $t \in T$. Moreover, if $\phi\colon A \to B$ is a ring homomorphism such that ϕ sends every element in ST to a unit in B, then it sends every element of S as well as every element of T to a unit in B. We therefore have a unique ring map $\psi_1\colon S^{-1}A \to B$ such that $\psi_1(f_1(a)) = \phi(a)$ for every $a \in A$. It follows that $\psi_1(f_1(t)) = \phi(t)$ is a unit for every $t \in T$. Using this and the fact that $(f_1(T)^{-1}(S^{-1}A) = T^{-1}(S^{-1}A)$ we get a unique map $\psi\colon (f_1(T))^{-1}(S^{-1}A) \to B$ such that $\psi \circ f_2 = \psi_1$. It follows that $\psi \circ f = \phi$. We have to show the uniqueness of ψ.

We make a quick observation before returning to the proof. Under ring homomorphisms, units get mapped to units, and the their multiplicative inverses to the multiplicative inverses of the images. In particular, if $s \in S$, then $f_2(f_1(s)^{-1}) = f(s)^{-1}$.

If $\psi'\colon T^{-1}A(S^{-1}A) \to B$ is another ring homomorphism such that $\psi \circ f = \phi$, then for $a \in A$, and $s \in S$,

$$\begin{aligned}\psi'(f_2(a/s)) = \psi'(f_2(f_1(s)^{-1})f(a)) =& \psi'(f(s)^{-1}f(a)) \\ &= \psi'(f(s))^{-1}\psi'(f(a)) \\ &= \phi(s)^{-1}\phi(a) \\ &= \psi_1(a/s).\end{aligned}$$

Thus $\psi' \circ f_2 = \psi_1$. By the universal property of $T^{-1}(S^{-1}A)$, this means $\psi' = \psi$.

2. **(a)** If $f, g \in A$ are such that $D(f) = D(g)$ as subsets X, then for $M \in \mathrm{Mod}_A$, $M_f = M_g$.

(b) Let f_α, $\alpha \in \mathbf{\Lambda}$, be a family of elements of A such that $\langle f_\alpha \rangle = A$. Suppose $s_\alpha \in M_{f_\alpha}$, $\alpha \in \mathbf{\Lambda}$ is a family of elements satisfying $s_\alpha/1 = s_\beta/1$ (as elements in $M_{f_\alpha f_\beta}$) for $\alpha, \beta \in \mathbf{\Lambda}$. Show that there exists a unique element $m \in M$ such that $m/1 \in M_{f_\alpha}$ equals s_α for $\alpha \in \mathbf{\Lambda}$.

Solution:

(a) It is clear from the description of saturation in the formula $(*)$ that the multiplicative systems $\{f^n \mid n \geq 0\}$ and $\{g^n \mid n \geq 0\}$ have the same saturation, namely

$$T = A \smallsetminus \bigcup_{\mathfrak{p} \in D(f)} \mathfrak{p} = A \smallsetminus \bigcup_{\mathfrak{p} \in D(g)} \mathfrak{p}.$$

(b) Let us first prove that such an $m \in M$ is unique. It is enough to prove that if $m/1 = 0 \in A_{f_\alpha}$ for every $\alpha \in \mathbf{\Lambda}$, then $m = 0$. Let $m \in M$ be such an element. We have non-negative integers n_α, one for each $\alpha \in \mathbf{\Lambda}$, such that $f_\alpha^{n_\alpha} m = 0$. Now $\langle f_\alpha^{n_\alpha} \mid \alpha \in \mathbf{\Lambda} \rangle = A$ (see the first proposition on page 3 of Lecture 11). There exist $\alpha_1, \dots, \alpha_e \in \mathbf{\Lambda}$ and $x_1, \dots, x_e \in A$ such that $\sum_{i=1}^{e} x_i f_{\alpha_i}^{n_{\alpha_i}} = 1$. Then

$$m = 1 \cdot m = \Big(\sum_{i=1}^{e} x_i f_{\alpha_i}^{n_{\alpha_i}}\Big) m = 0.$$

This proves uniqueness.

For the existence of m, let us use the quasi compactness of Spec(A). We have $\alpha_1, \ldots, \alpha_d \in \mathbf{\Lambda}$ such that with $g_i := f_{\alpha_i}$, we have $\mathrm{Spec}(A) = \cup_{i=1}^d D(g_i)$. Since $A_{g_i^e} = A_{g_i}$ for all $e \geq 1$ and $i = 1, \ldots, d$, replacing all the g_i by suitable powers, we may assume there exist $m_1, \ldots, m_d \in M$ such that

$$s_{\alpha_i} = m_i/g_i, \qquad i = 1, \ldots, d.$$

In Spec$(A_{g_i g_j})$ we have $m_i/g_i = m_j/g_j$, whence there exist $n_{ij} \geq 0$ such that $(g_i g_j)^{n_{ij}} g_j m_i = (g_i g_j)^{n_{ij}} g_i m_j$, $1 \leq i, j \leq d$. Set $n = \max_{1 \leq i,j \leq d} n_{ij}$. Then

$$(\dagger) \qquad g_j^{n+1} g_i^n m_i = g_i^{n+1} g_j^n m_j, \qquad 1 \leq i, j \leq d.$$

Now $\langle g_1^{n+1}, \ldots, g_d^{n+1} \rangle = A$, hence there exist $a_j \in A$ such that

$$\sum_{j=1}^d a_j g_j^{n+1} = 1.$$

In particular, using ($\dagger$), we get

$$g_i^n m_i = \Big(\sum_{j=1}^d a_j g_j^{n+1}\Big) g_i^n m_i = g_i^{n+1} \sum_{j=1}^d a_j g_j^n m_j.$$

Let $m = \sum_{j=1}^d a_j g_j^n m_j$. The relations $g_i^n m_i = g_i^{n+1} m$ established above shows that $m/1 = m_i/g_i$ in A_{g_i} for $i = 1, \ldots, d$.

We have yet to establish that in A_{f_α}, $m/1 = s_\alpha$ for every $\alpha \in \mathbf{\Lambda}$ (we have only established this for $\alpha = \alpha_i$, $i = 1, \ldots, d$). So suppose $\alpha \in \mathbf{\Lambda}$. Let $\sigma = m/1 \in A_{f_\alpha}$. For $i = 1, \ldots, d$, let $\bar{g}_i$ be the image of g_i in A_{f_α}. Then $\langle \bar{g}_1, \ldots, \bar{g}_d \rangle = A_{f_\alpha}$. The image of m in $(A_{f_\alpha})_{\bar{g}_i} = A_{f_\alpha g_i} = (A_{g_i})_{f_\alpha}$ is, by the transitivity of localization (i.e. part **(c)** of Problem **1**), the image of s_α in $A_{f_\alpha g_i} = (A_{f_\alpha})_{\bar{g}_i}$.Thus the image of $\sigma - s_\alpha$ in $A_{f_\alpha g_i}$ is zero for $i = 1, \ldots d$. By the argument given for the uniqueness of m in the first paragraph of this proof, we see that $\sigma = s_\alpha$.

3. Let $A \to B$ be a ring homomorphism.

(a) Show that $A \otimes_A M = M$ for all $M \in \mathrm{Mod}_A$.

(b) Let $M \in \mathrm{Mod}_A$. Show that $B \otimes_A M \in \mathrm{Mod}_B$, where the scalar multiplication is such that $b(b' \otimes m) = (bb') \otimes m$, $b, b' \in B$, $m \in M$.

(c) For $M, N \in \mathrm{Mod}_B$ and $T \in \mathrm{Mod}_A$, show that we have a the following Hom–$\otimes$ adjointness:

$$\mathrm{Hom}_B(M, \mathrm{Hom}_A(N, T)) \xrightarrow{\sim} \mathrm{Hom}_A(M \otimes_B N, T).$$

Solution:

(a) This is clear. The scalar multiplication map $S \colon A \times M \to M$ is by definition bilinear, and if $B \colon A \times M \to T$ is a bilinear map over A, then clearly $m \mapsto B(1, m)$ defines an A-linear map $\varphi \colon M \to T$ such that $\varphi \circ S = B$. Moreover if $u \colon M \to T$ is any A-map such that $u \circ S = B$, then $u(m) = u(S(1, m)) = B(1, m)$, whence $u = \varphi$.

(b) Let $b \in B$. The A-bilinear map $B \times M \to B \otimes_A M$ given by $(b', m) \mapsto (bb', m)$ gives us a well defined map A-linear map $\nu(b)\colon B \otimes_A M \to B \otimes_A M$ given by $\nu(b)(\sum_\alpha (b_\alpha \otimes m_\alpha)) = \sum_\alpha (bb_\alpha) \otimes m_\alpha$. It is clear, since multiplication is distributive and $\otimes$ is bilinear, that $\nu(b_1 + b_2) = \nu(b_1) + \nu(b_2)$. Moreover, $\nu(b)$, being an A-module map, respects addition in $B \otimes_A M$. Finally, $\nu(1)$ is obviously the identity map. Collecting these facts together, we see that $(b, \sum_\alpha b_\alpha \otimes m_\alpha) \mapsto \nu(b)(\sum_\alpha b_\alpha \otimes m_\alpha)$ defines a scalar product

$$B \times (B \otimes_A M) \longrightarrow B \otimes_A M.$$

(c) This is *mutatis mutandis* the proof given in class for $B = A$. Details left to you.

4. An A-module E is called *injective* if $\mathrm{Hom}_A(-, E)$ is an exact functor (i.e., it transforms exact sequences of A-modules into exact sequences of A-modules). Let $A \to B$ be a ring map and E an injective A-module. Show that $\mathrm{Hom}_A(B, E)$ is an injective B-module. [**Hint:** Use the above Hom–$\otimes$. adjointness. Note that we have already seen in class that $\mathrm{Hom}_A(B, E)$ is a B-module.]

Solution: First note that a sequence of B-modules

$$\cdots \to M^{i-1} \to M^i \to M^{i+1} \to \ldots$$

is exact if and only if when thought of as a sequence of A-modules it is exact.

Now

$$\mathrm{Hom}_B(-, \mathrm{Hom}_A(B, E)) \xrightarrow{\sim} \mathrm{Hom}_A(- \otimes_B B, E) = \mathrm{Hom}_A(-, E).$$

The last functor is exact, and we are done.