HW 3

Due date: Feb 17, 2022 (to be handed over in class)

Throughout, if A is a ring and $X = \operatorname{Spec} A$. The conventions used in lectures and the previous homework assignments remain in force. A reminder: if $f \in A$, then

$$D(f) := \{\mathfrak{p} \in X \mid f \notin \mathfrak{p}\}.$$

According to what was done (or will be done) in the lectures, $D(f)$ is homeomorphic to $\operatorname{Spec} A_f$ and in fact $D(f)$ is the homeomorphic image of $\operatorname{Spec} A_f$ under the natural map $\operatorname{Spec} A_f \to \operatorname{Spec} A = X$ induced by the localization map $A \to A_f$.

1. For a multiplicatively closed set S in A, we call the set

$$\overline{S} := \{a \in A \mid ab \in S \text{ for some } b \in A\}$$

the *saturation* of S. Let $f, g \in A$ and $M \in \mathrm{Mod}_A$. Fix a multiplicative system S.

- **(a)** Show that $\overline{S}$ is multiplicatively closed and that $\overline{S}^{-1} A = S^{-1} A$.
- **(b)** Let T be a multiplicative system such that $S \subset T \subset \overline{S}$. Show that $\overline{T} = \overline{S}$. In particular, conclude that $\overline{\overline{S}} = \overline{S}$.
- **(c)** Let T be another multiplicative system, and let ST be the multiplicative system $ST := \{st \in A \mid s \in S,\ t \in T\}$. Show that $(ST)^{-1} M = T^{-1}(S^{-1} M)$.
- **(d)** Show that if $d \geq 1$ then $M_{f^d} = M_f$.
- **(e)** Show that $(M_f)_g = M_{fg}$.

2.
- **(a)** If $f, g \in A$ are such that $D(f) = D(g)$ as subsets X, then for $M \in \mathrm{Mod}_A$, $M_f = M_g$.
- **(b)** Let f_α, $\alpha \in \mathbf{\Lambda}$, be a family of elements of A such that $\langle f_\alpha \rangle = A$. Suppose $s_\alpha \in M_{f_\alpha}$, $\alpha \in \mathbf{\Lambda}$ is a family of elements satisfying $s_\alpha/1 = s_\beta/1$ as elements in $M_{f_\alpha f_\beta}$) for $\alpha, \beta \in \mathbf{\Lambda}$. Show that there exists a unique element $m \in M$ such that $m/1 \in M_{f_\alpha}$ equals s_α for $\alpha \in \mathbf{\Lambda}$.

Remark: Part (a) gives us another way of seeing that $M_f = M_{f^d}$.

3. Let $A \to B$ be a ring homomorphism.

- **(a)** Show that $A \otimes_A M = M$ for all $M \in \mathrm{Mod}_A$.
- **(b)** Let $M \in \mathrm{Mod}_A$. Show that $B \otimes_A M \in \mathrm{Mod}_B$, where the scalar multiplication is such that $b(b' \otimes m) = (bb') \otimes m$, $b, b' \in B$, $m \in M$.
- **(c)** For $M, N \in \mathrm{Mod}_B$ and $T \in \mathrm{Mod}_A$, show that we have a the following Hom–$\otimes$ adjointness:

$$\mathrm{Hom}_B(M,\, \mathrm{Hom}_A(N,\, T)) \xrightarrow{\sim} \mathrm{Hom}_A(M \otimes_B N,\, T).$$

4. An A-module E is called *injective* if $\mathrm{Hom}_A(-,\, E)$ is an exact functor (i.e., it transforms exact sequences of A-modules into exact sequences of A-modules). Let $A \to B$ be a ring map and E an injective A-module. Show that $\mathrm{Hom}_A(B,\, E)$ is an injective B-module. [**Hint:** Use the above Hom–$\otimes$. adjointness. Note that we have already seen in class that $\mathrm{Hom}_A(B,\, E)$ is a B-module.]