HW-2 SOLUTIONS

Throughout, if A is a ring[1] then Mod_A denotes the category of A-modules. If $M, N \in \mathrm{Mod}_A$, we will often use the phrase "$f\colon M \to N$ is an A-map" as a shorthand for "$f\colon M \to N$ is a homomorphism of A-modules".

For an element $a \in A$ and an A-module M, $\mu_a\colon M \to M$ will denote the A-map $x \mapsto ax$. The map μ_a is often called the "multiplication by a" map.

Direct Limits. Let $\mathbf{\Lambda}$ be a partially ordered set, with partial order $\prec$. We say $\mathbf{\Lambda}$ is a *directed set* if given α and β in $\mathbf{\Lambda}$, there exists $\gamma \in \mathbf{\Lambda}$ such that $\alpha \prec \gamma$ and $\beta \prec \gamma$. Now let A be a ring and $(\mathbf{\Lambda}, \prec)$ a directed set. A *direct system over the directed set* $\mathbf{\Lambda}$ of A-modules is a collection of A-modules $\{M_\alpha \mid \alpha \in \mathbf{\Lambda}\}$ together with A-maps $\mu_{\alpha\beta}\colon M_\alpha \to M_\beta$, one for every pair of indices (α, β) such that $\alpha \prec \beta$ satisfying the following relations:

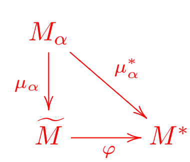
(i) $\mu_{\alpha\alpha} = id_{M_\alpha}$, $\alpha \in \mathbf{\Lambda}$;
(ii) $\mu_{\alpha\gamma} = \mu_{\beta\gamma} \circ \mu_{\alpha\beta}$ whenever $\alpha \prec \beta \prec \gamma$.

One often writes (M_α) or $(M_\alpha)_{\alpha \in \mathbf{\Lambda}}$ for the collection $\{M_\alpha \mid \alpha \in \mathbf{\Lambda}\}$ as well as for the direct system $\mathbf{M} = (M_\alpha, \mu_{\alpha\beta})$, suppressing the $\mu_{\alpha\beta}$. Now suppose $\mathbf{M} = (M_\alpha)$ is a direct system over $\mathbf{\Lambda}$. The[2] *direct limit* of $\mathbf{M}$ is an A-module $\widetilde{M}$ together with a collection of A-maps $\mu_\alpha\colon M_\alpha \to \widetilde{M}$, one for each $\alpha \in \mathbf{\Lambda}$, such that $\mu_\alpha = \mu_\beta \circ \mu_{\alpha\beta}$ for every $\alpha \prec \beta$; this data satisfying the following condition: If $T \in \mathrm{Mod}_A$, and one has A-maps $\nu_\alpha\colon M_\alpha \to T$, $\alpha \in \mathbf{\Lambda}$ satisfying $\nu_\alpha = \nu_\beta \circ \mu_{\alpha\beta}$ whenever $\alpha \prec \beta$, then there exists a unique A-map $\nu\colon \widetilde{M} \to T$ such that $\nu_\alpha = \nu \circ \mu_\alpha$ for every $\alpha \in \mathbf{\Lambda}$. (We will soon change notation, and use the symbol $\varinjlim_{\alpha} M_\alpha$ for the direct limit.)

In what follows assume that a direct limit always exists for a direct system, and for definiteness, fix one for each direct system.

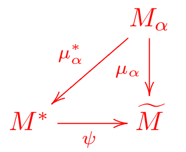
1. Show that $(\widetilde{M}, \mu_\alpha)_\alpha$ is unique up to unique isomorphism. In other words, show that is $(M^*, \mu^*_\alpha)_\alpha$ is another pair enjoying the same universal property that $(\widetilde{M}, \mu_\alpha)_\alpha$ does, then there is a unique isomorphism $\varphi\colon \widetilde{M} \xrightarrow{\sim} M^*$ such that $\mu^*_\alpha = \varphi \circ \mu_\alpha$.

Solution: The universal property of $(\widetilde{M}, \mu_\alpha)$ gives us a unique A-map $\varphi\colon \widetilde{M} \to M^*$ such that the following diagram commutes for every α:

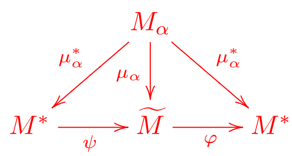


[1]According to our conventions, commutative and with identity.
[2]The definite article "the" will be justified in the problems below.

Similarly, the universal property for $(M^*,\ \mu^*_\alpha)$ gives us a unique A-map $\psi\colon M^* \to \widetilde{M}$ such that the following diagram commutes for each α:



Combining the two we have commutative diagrams, one for each α:



Thus $(\varphi\circ\psi)\circ\mu^*_\alpha = \mu^*_\alpha$ for every α. On the other hand $\mathbf{1}_{M^*}\circ\mu^*_\alpha = \mu^*_\alpha$. for every α. By the universal property of $(M^*,\ \mu^*_\alpha)$ there is exactly one solution for u in the system of equations $u\circ\mu^*_\alpha = \mu^*_\alpha$, $\alpha \in \mathbf{\Lambda}$, and hence $\varphi\circ\psi = \mathbf{1}_{M^*}$.

By symmetry $\psi\circ\varphi = \mathbf{1}_{\widetilde{M}}$. $\square$

Since "the" direct limit is unique up to unique isomorphism, we will regard it as unique, and write

$$\varinjlim_{\alpha} M_\alpha$$

for the above direct limit. As with so much of mathematics (think groups), we will write $\varinjlim_{\alpha} M_\alpha$ for the module $\widetilde{M}$ as well as the data $(\widetilde{M},\ \mu_\alpha)_\alpha$.

For Problem 2 below, let $(M_\alpha)_{\alpha\in\mathbf{\Lambda}}$ be a direct system of A-modules. Let $\coprod_{\alpha\in\mathbf{\Lambda}} M_\alpha$ be the disjoint union of the M_α, i.e. $\coprod_{\alpha\in\mathbf{\Lambda}} M_\alpha = \bigcup_{\alpha\in\mathbf{\Lambda}}\{\alpha\}\times M_\alpha$. For $(\alpha,\ m_\alpha)$, $(\beta,\ m_\beta)$ in $\coprod_\alpha M_\alpha$, write $(\alpha, m_\alpha) \sim (\beta, m_\beta)$ if there exists a $\gamma \succ \alpha, \beta$ such that $\mu_{\alpha\gamma}(m_\alpha) = \mu_{\beta\gamma}(m_\beta)$. It is easy to see that $\sim$ is an equivalence relation on $\coprod_\alpha M_\alpha$. Assume this in what follows (and prove it for yourself, but don't show me the work). Let $[\alpha, m_\alpha]$ denote the equivalence class of (α, m_α). One can define an A-module structure on $M = (\coprod_\alpha M_\alpha)/\!\sim$ by setting

$$[\alpha, m_\alpha] + [\beta, m_\beta] := [\gamma, \mu_{\alpha\gamma}(m_\alpha) + \mu_{\beta\gamma}(m_\beta)]$$

for any $\gamma \succ \alpha, \beta$, and by setting

$$a[\alpha, m_\alpha] := [\alpha, am_\alpha]$$

for $a \in A$. It is easy to see (and again – you don't have to show this to me, but do work it out for yourself) that the addition and scalar multiplication on M given above is well-defined and defines an A-module structure on M. You may use all these easily proven facts in what follows.

2. For $\alpha \in \mathbf{\Lambda}$, let $\nu_\alpha\colon M_\alpha \to M$ be the map $m_\alpha \mapsto [\alpha,\ m_\alpha]$.

- **(a)** Show that each ν_α is an A-map.
- **(b)** Show that for $\alpha \prec \beta$, $\nu_\alpha = \nu_\beta\circ\mu_{\alpha\beta}$.
- **(c)** Show that if $\nu_\alpha(x) = 0$ for some $x \in M_\alpha$, then for some $\beta \succ \alpha$, $\mu_{\alpha\beta}(x) = 0$.
- **(d)** Show that the unique map $\nu\colon \varinjlim_{\alpha} M_\alpha \to M$, arising from the universal property of direct limits, is an isomorphism.

Solution: Parts (a), (b) and (c) are straightforward, so I will concentrate on (d).

Suppose $[\alpha,\, m_\alpha] = [\beta,\, m_\beta]$. Then there exists $\gamma \succ \alpha, \beta$ such that $\mu_{\alpha\gamma}(m_\alpha) = \mu_{\beta\gamma}(m_\beta)$. Denote the common element in M_γ defined by the this equality by m_γ. Now $\mu_\alpha(m_\alpha) = \mu_\gamma(\mu_{\alpha\gamma}(m_\alpha)) = \mu_\gamma(m_\gamma)$. The same reasoning shows that $\mu_\beta(m_\beta) = \mu_\gamma(m_\gamma)$. In particular $\mu_\alpha(m_\alpha) = \mu_\beta(m_\beta)$. We have thus proved that the map

$$\theta\colon M \to \widetilde{M}$$

given by

$$\theta([\alpha,\, m_\alpha]) = \mu_\alpha(m_\alpha),$$

is well-defined. It is easy to see this is an A-map (though you are expected to spell this out in greater detail than I have). Now for $m_\alpha \in M_\alpha$, we have $\theta(\nu_\alpha(m_\alpha)) = \theta([\alpha,\, m_\alpha]) = \mu_\alpha(m_\alpha)$, whence $\theta\circ\nu_\alpha = \mu_\alpha$ for every index α. This means that for $\alpha \in \mathbf{\Lambda}$, we have $(\theta\circ\nu)\circ\mu_\alpha = \theta\circ(\nu\circ\mu_\alpha) = \theta\circ\nu_\alpha = \mu_\alpha$. Thus $u = \theta\circ\nu$ is a solution to the system of equations $u\circ\mu_\alpha = \mu_\alpha$, $\alpha \in \mathbf{\Lambda}$. By the universal property of direct limits, there is only solution to this system, namely $u = \mathbf{1}_{\varinjlim_\alpha M_\alpha}$. It follows that $\theta\circ\nu$ is the identity map on $\varinjlim_\alpha M_\alpha$.

On the other hand, for $[\alpha,\, m_\alpha] \in M$, we have

$$\begin{aligned}\nu(\theta([\alpha,\, m_\alpha])) = \nu(\mu_\alpha(m_\alpha)) &= (\nu\circ\mu_\alpha)(m_\alpha)\\ &= \nu_\alpha(m_\alpha)\\ &= [\alpha,\, m_\alpha].\end{aligned}$$

Thus $\nu\circ\theta = \mathbf{1}_M$. □

Tensor products. Let A be a ring and $M, N, T \in \mathrm{Mod}_A$. A *bilinear map of A-modules* from $M\times N$ to T is a map $B\colon M\times N \to T$ such that for every $m, m' \in M$, $n, n' \in N$ and $a \in A$ we have

(i) $B(m+m',\, n) = B(m, n) + B(m',\, n)$;
(ii) $B(m, n+n') = B(m, n) + B(m, n')$;
(iii) $B(am,\, n) = B(m,\, an) = aB(m, n)$.

The *tensor product of M and N over A* is an A-module $M\otimes_A N$ together with a bilinear map of A-modules $B_u\colon M\times N \to M\otimes_A N$ such that given a bilinear map of A-modules $B\colon M\times N \to T$, there exists a unique an A-module map $\psi\colon M\otimes_A N \to A$ satisfying $B = \psi\circ B_u$. In what follows, assume that such a pair $(M\otimes_A N,\, B_u)$ exists, and fix one such pair for definiteness.

3. Show that $(M\otimes_A N,\, B_u)$ is unique up to unique isomorphism, i.e. if $(M*N, B^*)$ is another pair, with $B^*\colon M\times N \to M*N$ a bilinear map of A-modules enjoying the universal property that $(M\otimes_A N,\, B_u)$ does, then there is a unique isomorphism of A-modules $\varphi\colon M\otimes_A N \xrightarrow{\sim} M*N$ such that $\varphi\circ B_u = B^*$.

Solution: The universal property of $(M\otimes_A N,\, B_u)$ give us a unique map $\varphi\colon M\otimes_A N \xrightarrow{\sim} M*N$ such that $\varphi\circ B_u = B^*$. By symmetry, the universal property of $M*N,\, B^*)$ gives us unique map $\psi\colon M*N \to M\otimes_A B$ such that $\psi\circ B^* = B_u$. Thus $(\psi\circ\varphi)\circ B_u = B_u$ and $(\varphi\circ\psi)\circ B^* = B^*$. However, the by the universal property of $(M\otimes_A N, B_u)$ and $M*N,\, B^*)$, the equations $x\circ B_u = B_u$ $y\circ B^* = B^*$ admit exactly one solution each, namely $x = \mathbf{1}_{M\otimes_A N}$ and $y = \mathbf{1}_{M*N}$. Thus $\psi\circ\varphi$ and $\varphi\circ\psi$ are both identity maps. □

4. Write $m \otimes n$ for $B_u(m, n) \in M \otimes_A N$. Show that $M \otimes_A N$ is generated as an A-module by elements of the form $m \otimes n$. You are expected to use the universal property of tensor products and not any construction you might have seen. [**Hint:** Let U be the submodule of $M \otimes_A N$ generated by the $m \otimes n$. See if you can prove that U has the required universal property.]

Solution: Clearly $U \supset B_u(M \times N)$. Let $B' \colon M \times N \to U$ be the map induced by B_u, i.e. $B'(m, n) = B_u(m, n)$ for $(m, n) \in M \times N$. It is quite evident that (U, B') has the required universal property of tensor products.[3] If the incliusion $B \subset M \otimes_A N$ is denoted by $i \colon U \to M \otimes_A N$, then $i \circ B' = B_u$. By the universal property of B_u, we also have a map $\varphi \colon M \otimes_A N \to U$ usch that $\varphi \circ B_u = B'$. By a now familiar argument, $i \circ \varphi$ is the identity on $M \otimes_A N$ and $\varphi \circ i$ is the identity on U. This means the inclusion $i \colon U \hookrightarrow M \otimes_A N$ is an isomorphism, i.e. U is equal to $M \otimes_A N$. □

5. Show using the universal property of tensor products and the universal property of localizations that if S is a multiplicative system in a ring A, and $M \in \mathrm{Mod}_A$, then $S^{-1}M = S^{-1}A \otimes_A M$. [**Hint:** Define a bilinear map $S^{-1}A \times M \to S^{-1}M$ and show that it has the universal property for tensor products. To do that, you may need to verify that that if one has a bilinear map $M \times N \to T$ then on the submodule of T generated by image of $M \times N$, μ_s is an isomorphism for every $s \in S$.]

Solution: Let $m \in M$ and suppose a/s and a'/s' are two representations of an element $x \in S^{-1}A$. We claim that $\frac{am}{s} = \frac{a'm}{s'}$. The equality $a/s = a'/s'$ gives us an element $t \in S$ such that $ts'a = tsa'$. This means $ts'am = tsa'm$, whence $(am)/s = (a'm)/s'$ as asserted. This means that the map $B^* \colon S^{-1}A \times M \to S^{-1}M$ given by the formula

$$B^*\Big(\frac{a}{s}, m\Big) = \frac{am}{s}$$

is well-defined. It is easy to see that B^* is bilnear over A.

Next suppose $B \colon M \times N \to T$ is a bilnear map over A. Define $\varphi \colon S^{-1}M \to T$ by the rule $\varphi(m/s) = B(1/s, m)$. We have to show (among other things) that φ is well-defined. To that end, suppose $m/s = m'/s'$. We have an element $t \in S$ such that $ts'm = tsm'$. By the A-bilinearity of B, we get

$$\begin{aligned} B\Big(\frac{1}{s}, m\Big) = B\Big(\frac{ts'}{ts's}, m\Big) &= B\Big(\frac{1}{ts's}, ts'm\Big) \\ &= B\Big(\frac{1}{ts's}, tsm'\Big) \\ &= B\Big(\frac{ts}{ts's}, m'\Big) \\ &= B\Big(\frac{1}{s'}, m'\Big). \end{aligned}$$

This proves φ is well-defined. Moreover, for $m, m' \in M$, $s, s' \in S$, and $a \in A$, we have $\varphi(\frac{m}{s} + a\frac{m'}{s'}) = \varphi(\frac{s'm + sam'}{ss'}) = B(\frac{1}{ss'}, s'm + sam') = B(\frac{1}{ss'}, s'm) + aB(\frac{1}{ss'}, sm') = B(\frac{1}{s}, m) + aB(\frac{1}{s'}, m') = \varphi(\frac{m}{s}) + a\varphi(\frac{m'}{s'})$. In other words, φ is A-linear. Now $\varphi(B^*(a/s, m)) = \varphi((am)/s) = B(1/s, am) = B(a/s, m)$. Thus $\varphi \circ B^* = B$.

[3] Of course, you are expected to provide more details

It remains to show that φ is the unique map with the above property, i.e. we have to show that if $\psi\colon S^{-1}M \to T$ is an A-map such that $\psi \circ \psi = B$, then $\psi = \varphi$. Let ψ be such a map. Then $\psi(m/s) = \psi(B^*(1/s,\, m) = B(1/s,\, m) = \varphi(m/s)$. □

Localization and direct limits. Certain localizations can be regarded as direct limits.

6. Let A be a ring, f an element of A, and T an A-module. Define a direct system $\mathbf{T} = (T_n, \mu_{m,n})$, where the indices vary over non-negative integers (with its natural structure as a directed set), by $T_n = T$ for all $n \geq 0$; and for $m \leq n$, $\mu_{m,n}\colon T_m \to T_n$ is the map $x \mapsto f^{n-m}x$. Show that $\varinjlim_n T_n = T_f$, where T_f is the localization of T at the multiplicative system $\{f^n \mid n \geq 0\}$.

Solution: Let $\mu_n\colon T_n \to T_f$ be the map $x \mapsto x/f^n$. It is clear that $\mu_n \circ \mu_{m,n} = \mu_m$ for $0 \leq m \leq n$. Next suppose $Y \in \mathrm{Mod}_A$ and we have a family of A-maps, $\nu_n\colon T_n \to Y$, one for each $n \geq 0$, such that $\nu_n \circ \mu_{m,n} = \nu_m$ for $0 \leq m \leq n$. If $\psi\colon T_f \to Y$ is an A-map such that $\nu_n = \psi \circ \mu_n$ for every $n \geq 0$, then $\psi(x/f^n) = \psi(\mu_n(x)) = \nu_n(x)$, which means there is at most one ψ such that the relation $\nu_n = \psi \circ \mu_n$ holds for every $n \geq 0$.

We will now show that for Y and $\{\nu_n\}_n$ as above, such a map ψ exists. This is equivalent to showing that $x/f^n \mapsto \nu_n(x)$ is a well defined A-map on T_f. To that end, suppose $x/f^n = x'/f^m$, and for definiteness, assume $m \leq n$. There exists $d \geq 0$ such that $f^d(f^m x - f^n x') = 0$. In other words, we have $f^{m+d}x = f^{n+d}x'$. Regard x as an element of T_n and x' as an element of T_m. We have just proven that $\mu_{n,m+n+d}(x) = \mu_{m,m+n+d}(x')$. It follows that

$$\nu_n(x) = \nu_{m+n+d}(\mu_{n,m+n+d}(x)) = \nu_{m+n+d}(\mu_{m,m+n+d}(x')) = \nu_m(x').$$

Thus the map $x/f^n \mapsto \nu_n(x)$ is a well-defined map on T_f taking values in Y.

Let $\psi\colon T_f \to Y$ denote this map. It is clear that $\psi \circ \mu_n = \nu_n$ for every $n \geq 0$.

Next we will show that ψ is an A-map. This is seen from the following computation for $x, x' \in T$, $n, m \geq 0$, and $a \in A$:

$$\begin{aligned}
\psi\left(\frac{x}{f^n} + a\frac{x'}{f^m}\right) = \psi\left(\frac{f^m x + a f^n x'}{f^{m+n}}\right) &= \nu_{m+n}(f^m x + a f^n x') \\
&= \nu_{m+n}(f^m x) + a\nu_{m+n}(f^n x') \\
&= \psi\left(\frac{f^m x}{f^{m+n}}\right) + a\psi\left(\frac{f^n x'}{f^{m+n}}\right) \\
&= \psi\left(\frac{x}{f^n}\right) + a\psi\left(\frac{x'}{f^m}\right).
\end{aligned}$$

It follows that $(T_f,\, \mu_n)$ has the universal property required of a direct limit. □