HW 2

Due date: Feb 8, 2022

Throughout, if A is a ring[1] then Mod_A denotes the category of A-modules. If $M, N \in \mathrm{Mod}_A$, we will often use the phrase "$f\colon M \to N$ is an A-map" as a shorthand for "$f\colon M \to N$ is a homomorphism of A-modules".

For an element $a \in A$ and an A-module M, $\mu_a\colon M \to M$ will denote the A-map $x \mapsto ax$. The map μ_a is often called the "multiplication by a" map.

This homework assignment is mainly about universal properties. We mentioned in class that localization could be defined via the *universal property of localzation*. In this approach, the definition we gave becomes an existence theorem. In this homework assignment we give two other examples of objects defined via their universal properties. We will show that these objects exist in the lectures by "constructing" them.

Direct Limits. Let $\mathbf{\Lambda}$ be a partially ordered set, with partial order $\prec$. We say $\mathbf{\Lambda}$ is a *directed set* if given α and β in $\mathbf{\Lambda}$, there exists $\gamma \in \mathbf{\Lambda}$ such that $\alpha \prec \gamma$ and $\beta \prec \gamma$. Now let A be a ring and $(\mathbf{\Lambda}, \prec)$ a directed set. A *direct system over the directed set* $\mathbf{\Lambda}$ of A-modules is a collection of A-modules $\{M_\alpha \mid \alpha \in \mathbf{\Lambda}\}$ together with A-maps $\mu_{\alpha\beta}\colon M_\alpha \to M_\beta$, one for every pair of indices (α, β) such that $\alpha \prec \beta$ satisfying the following relations:

(i) $\mu_{\alpha\alpha} = id_{M_\alpha}$, $\alpha \in \mathbf{\Lambda}$;
(ii) $\mu_{\alpha\gamma} = \mu_{\beta\gamma} \circ \mu_{\alpha\beta}$ whenever $\alpha \prec \beta \prec \gamma$.

One often writes (M_α) or $(M_\alpha)_{\alpha \in \mathbf{\Lambda}}$ for the collection $\{M_\alpha \mid \alpha \in \mathbf{\Lambda}\}$ as well as for the direct system $\mathbf{M} = (M_\alpha, \mu_{\alpha\beta})$, suppressing the $\mu_{\alpha\beta}$. Now suppose $\mathbf{M} = (M_\alpha)$ is a direct system over $\mathbf{\Lambda}$. The[2] *direct limit* of $\mathbf{M}$ is an A-module $\widetilde{M}$ together with a collection of A-maps $\mu_\alpha\colon M_\alpha \to \widetilde{M}$, one for each $\alpha \in \mathbf{\Lambda}$, such that $\mu_\alpha = \mu_\beta \circ \mu_{\alpha\beta}$ for every $\alpha \prec \beta$; this data satisfying the following condition: If $T \in \mathrm{Mod}_A$, and one has A-maps $\nu_\alpha\colon M_\alpha \to T$, $\alpha \in \mathbf{\Lambda}$ satisfying $\nu_\alpha = \nu_\beta \circ \mu_{\alpha\beta}$ whenever $\alpha \prec \beta$, then there exists a unique A-map $\nu\colon \widetilde{M} \to T$ such that $\nu_\alpha = \nu \circ \mu_\alpha$ for every $\alpha \in \mathbf{\Lambda}$. (We will soon change notation, and use the symbol $\varinjlim_\alpha M_\alpha$ for the direct limit.)

In what follows assume that a direct limit always exists for a direct system, and for definiteness, fix one for each direct system.

1. Show that $(\widetilde{M}, \mu_\alpha)_\alpha$ is unique up to unique isomorphism. In other words, show that is $(M^*, \mu^*_\alpha)_\alpha$ is another pair enjoying the same universal property that $(\widetilde{M}, \mu_\alpha)_\alpha$ does, then there is a unique isomorphism $\varphi\colon \widetilde{M} \xrightarrow{\sim} M^*$ such that $\mu^*_\alpha = \varphi \circ \mu_\alpha$.

Since "the" direct limit is unique up to unique isomorphism, we will regard it as unique, and write

$$\varinjlim_\alpha M_\alpha$$

[1] According to our conventions, commutative and with identity.
[2] The definite article "the" will be justified in the problems below.

for the above direct limit. As with so much of mathematics (think groups), we will write $\varinjlim_{\alpha} M_\alpha$ for the module $\widetilde{M}$ as well as the data $(\widetilde{M},\, \mu_\alpha)_\alpha$.

For Problem 2 below, let $(M_\alpha)_{\alpha\in\mathbf{\Lambda}}$ be a direct system of A-modules. Let $\coprod_{\alpha\in\mathbf{\Lambda}} M_\alpha$ be the disjoint union of the M_α, i.e. $\coprod_{\alpha\in\mathbf{\Lambda}} M_\alpha = \bigcup_{\alpha\in\mathbf{\Lambda}}\{\alpha\}\times M_\alpha$. For $(\alpha,\, m_\alpha)$, $(\beta,\, m_\beta)$ in $\coprod_\alpha M_\alpha$, write $(\alpha, m_\alpha) \sim (\beta, m_\beta)$ if there exists a $\gamma \succ \alpha, \beta$ such that $\mu_{\alpha\gamma}(m_\alpha) = \mu_{\beta\gamma}(m_\beta)$. It is easy to see that $\sim$ is an equivalence relation on $\coprod_\alpha M_\alpha$. Assume this in what follows (and prove it for yourself, but don't show me the work). Let $[\alpha, m_\alpha]$ denote the equivalence class of (α, m_α). One can define an A-module structure on $M = (\coprod_\alpha M_\alpha)/\!\sim$ by setting

$$[\alpha, m_\alpha] + [\beta, m_\beta] := [\gamma, \mu_{\alpha\gamma}(m_\alpha) + \mu_{\beta\gamma}(m_\beta)]$$

for any $\gamma \succ \alpha, \beta$, and by setting

$$a[\alpha, m_\alpha] := [\alpha, a m_\alpha]$$

for $a \in A$. It is easy to see (and again – you don't have to show this to me, but do work it out for yourself) that the addition and scalar multiplication on M given above is well-defined and defines an A-module structure on M. You may use all these easily proven facts in what follows.

2. For $\alpha \in \mathbf{\Lambda}$, let $\nu_\alpha \colon M_\alpha \to M$ be the map $m_\alpha \mapsto [\alpha,\, m_\alpha]$.
- **(a)** Show that each ν_α is an A-map.
- **(b)** Show that for $\alpha \prec \beta$, $\nu_\alpha = \nu_\beta \circ \mu_{\alpha\beta}$.
- **(c)** Show that if $\nu_\alpha(x) = 0$ for some $x \in M_\alpha$, then for some $\beta \succ \alpha$, $\mu_{\alpha\beta}(x) = 0$.
- **(d)** Show that the unique map $\nu \colon \varinjlim_{\alpha} M_\alpha \to M$, arising from the universal property of direct limits, is an isomorphism.

Tensor products. Let A be a ring and $M, N, T \in \mathrm{Mod}_A$. A *bilinear map of A-modules* from $M \times N$ to T is a map $B \colon M \times N \to T$ such that for every $m, m' \in M$, $n, n' \in N$ and $a \in A$ we have

- (i) $B(m + m',\, n) = B(m, n) + B(m',\, n)$;
- (ii) $B(m, n + n') = B(m, n) + B(m, n')$;
- (iii) $B(am,\, n) = B(m,\, an) = aB(m, n)$.

The *tensor product of M and N over A* is an A-module $M \otimes_A N$ together with a bilinear map of A-modules $B_u \colon M \times N \to M \otimes_A N$ such that given a bilinear map of A-modules $B \colon M \times N \to T$, there exists a unique an A-module map $\psi \colon M \otimes_A N \to A$ satisfying $B = \psi \circ B_u$. In what follows, assume that such a pair $(M \otimes_A N,\, B_u)$ exists, and fix one such pair for definiteness.

3. Show that $(M \otimes_A N,\, B_u)$ is unique up to unique isomorphism, i.e. if $(M * N, B^*)$ is another pair, with $B^* \colon M \times N \to M * N$ a bilinear map of A-modules enjoying the universal property that $(M \otimes_A N,\, B_u)$ does, then there is a unique isomorphism of A-modules $\varphi \colon M \otimes_A N \xrightarrow{\sim} M * N$ such that $\varphi \circ B_u = B^*$.

4. Write $m \otimes n$ for $B_u(m,\, n) \in M \otimes_A N$. Show that $M \otimes_A N$ is generated as an A-module by elements of the form $m \otimes n$. You are expected to use the universal property of tensor products and not any construction you might have seen. [**Hint:** Let U be the submodule of $M \otimes_A N$ generated by the $m \otimes n$. See if you can prove that U has the required universal property.]

5. Show using the universal property of tensor products and the universal property of localizations that if S is a multiplicative system in a ring A, and $M \in \mathrm{Mod}_A$, then $S^{-1}M = S^{-1}A \otimes_A M$. [**Hint:** Define a bilinear map $S^{-1}A \times M \to S^{-1}M$ and show that it has the universal property for tensor products. To do that, you may need to verify that that if one has a bilinear map $M \times N \to T$ then on the submodule of T generated by image of $M \times N$, μ_s is an isomorphism for every $s \in S$.]

Localization and direct limits. Certain localizations can be regarded as direct limits.

6. Let A be a ring, f an element of A, and T an A-module. Define a direct system $\mathbf{T} = (T_n, \mu_{m,n})$, where the indices vary over non-negative integers (with its natural structure as a directed set), by $T_n = T$ for all $n \geq 0$; and for $m \leq n$, $\mu_{m,n} \colon T_m \to T_n$ is the map $x \mapsto f^{n-m}x$. Show that $\varinjlim_n T_n = T_f$, where T_f is the localization of T at the multiplicative system $\{f^n \mid n \geq 0\}$.