SOLUTIONS OF HOMEWORK 1

These are brief solutions. Occasionally there might be a detailed solution, especially if a subtle point needs clarification.

Localization as a functor. In what follows A is a ring, and $S \subset A$ a multiplicative system. If $M \in \mathrm{Mod}_A$, we write

$$i_M \colon M \to S^{-1}M \tag{\#}$$

for the localization map $m \mapsto m/1$.

For an A-map $f\colon M \to N$ we define a map

$$S^{-1}f\colon S^{-1}M \longrightarrow S^{-1}N$$

by the rule $m/s \mapsto f(m)/s$.

1. Show the following:
- **(a)** $S^{-1}f$ is well-defined.
- **(b)** If $g\colon L \to M$ is a second map, then $S^{-1}(f \circ g) = (S^{-1}f) \circ (S^{-1}g)$.
- **(c)** If f is injective, so is $S^{-1}f$.
- **(d)** If f is surjective, so is $S^{-1}f$.

Solution: Parts (a) and (b) are straightforward.
(c) Suppose $(S^{-1}f)(m/s) = 0$, with $m \in M$ and $s \in S$. Then there exists an element $t \in S$ such that $tf(m) = 0$, i.e. $f(tm) = 0$. Since f is injective, this means $tm = 0$, which in turn means $m/s = 0$. □
(d) Let x/s be an element of $S^{-1}(N)$ with $x \in N$ and $s \in S$. Since f is surjective, there exists $m \in M$ such that $f(m) = x$ whence $(S^{-1}f)(m/s) = x/s$. □

2. Let L be an A-submodule of M. Regard $S^{-1}L$ as a submodule of $S^{-1}M$ via the part **(c)** of Problem **1**. Show that the map

$$S^{-1}(M/L) \longrightarrow (S^{-1}M)/(S^{-1}L)$$

given by $\frac{m+L}{s} \mapsto \frac{m}{s} + S^{-1}L$, is an isomorphism. (In what follows, and for the rest of the course, we will identify (without comment) $S^{-1}(M/L)$ with $(S^{-1}M)/(S^{-1}L)$ via the above isomorphism.)

Solution: Let us first verify that the given map is well-defined. Suppose, in an obvious notation, $(m+L)/s = (m'+L)/s'$. This means there exists $t \in S$ such that $t(s'm - sm' + L) = L$. In other words $ts'm - tsm' \in L$. Let $x = ts'm - tsm'$, whence $x/(tss') = m/s - m'/s$ as elements of $S^{-1}M$. Using the identification in part **(c)** of Problem **1**, this means that $m/s - m'/s' \in S^{-1}L$, i.e. $m/s + S^{-1}L = m'/s' + S^{-1}L$.

The given map is clearly surjective from its definition. Now suppose the image of $(m+L)/s = 0$, i.e. $m/s \in S^{-1}L$. Then $m/s = x/t$ for some $x \in L$ and $t \in S$. It follows that there is $t' \in S$ such that $t'(tm - sx) = 0$ in M. In particular $t'tm \in L$. Now $(m+L)/s = t't(m+L)/(t'ts) = (t'tm + L)/(t'ts) = 0$ since $t'tm + L$ is the zero element of M/L. □

3. Let $\ker(f)$, $\mathrm{im}(f)$, and $\mathrm{coker}(f)$ denote the kernel, image, and cokernel of f respectively. Show that

(a) $S^{-1}\ker(f) = \ker(S^{-1}f)$, where we regard both sides as submodules of $S^{-1}M$;

(b) $S^{-1}\mathrm{im}(f) = \mathrm{im}(S^{-1}f)$;

(c) $S^{-1}\mathrm{coker}(f) = \mathrm{coker}(S^{-1}f)$.

Solution: (a) Let $K = \ker f$ and $K_S = \ker(S^{-1}f)$. Let $x/s \in S^1K$. It is clear that $(S^{-1}f)(x/s) = f(x)/s = 0$. Thus $S^{-1}K \subset K_S$. Conversely, suppose $m/s \in K_S$. Then $f(m)/s = 0$, whence there exists $t \in S$ such that $tf(m) = 0$. Thus $tm \in K$, whence $m/s = (tm)/(ts) \in S^{-1}K$. □

(b) We regard $S^{-1}\mathrm{im}(f)$ and $\mathrm{im}(S^{-1}f)$ as submodules of $S^{-1}N$ via our earlier results. Let $H = \mathrm{im}(f)$ and $H_S = \mathrm{im}(S^{-1}f)$. Let $x/s \in S^{-1}H$ with $x \in H$. Since H is the image of f, we have $x = f(m)$ for some $m \in M$. Since $x/s = f(m)/s = (S^{-1}f)(m/s)$, we see that $S^{-1}H \subset H_S$. For the converse, suppose $n/s \in H_S \subset S^{-1}N$. Since $H_S = \mathrm{im}(S^{-1}f)$ we have $m/t \in S^{-1}M$ such that $f(m)/t = n/s$. From this we deduce that there exists $t' \in S$ such that $t'sf(m) = t'tn$, i.e. $t'tn \in H = \mathrm{im}(f)$. Since $n/s = (t'tn)/(t'ts) \in S^{-1}H$, we are done. □

(c) As above, let H be the image of f. Then this problem is really the problem of showing that $S^{-1}(N/H) = (S^{-1}N)/S^{-1}H$ with all the identifications made earlier. We are now reduced to Problem **2**. □

Localization of prime ideals. As before, A is a ring and $S \subset A$ a multiplicative system. Note that by part **(c)** of Problem **1**, if I is an ideal of A then $S^{-1}I$ can be regarded as an ideal of $S^{-1}A$. In what follows, we will so regard it.

If $f\colon A \to B$ is a ring homomorphism, and I an ideal in A, then IB will denote the ideal in B generated by the image of I in B. In other words, IB consists of finite sums of the form $\sum_\alpha b_\alpha f(x_\alpha)$, where the x_α lie in I and the b_α in B.

4. Let I be an ideal of A. Show that $S^{-1}I = S^{-1}A$ if and only if $S \cap I \neq \emptyset$.

Solution: Note that $S^{-1}I = S^{-1}A \iff 1/1 \in S^{-1}I \iff (\exists\, x \in I$ and $s \in S$ such that $1/1 = x/s) \iff (\exists\, t \in S$, $s \in S$ and $x \in I$ such that $st = tx)$. It is clear from the above equivalences that if $S^{-1}I = S^{-1}A$, then $S \cap I \neq \emptyset$, since $st = tx \in A \cap I$. Conversely, if $s \in S \cap I$, then pick $t = 1$ and $x = s$ in the last statement in the chain of equivalences above. □

5. Let $\mathfrak{p}$ be a prime ideal of A such that $S \cap \mathfrak{p} = \emptyset$. Show that $S^{-1}\mathfrak{p}$ is a prime ideal of $S^{-1}A$.

Solution: Let $(a/s)(a'/s') \in S^{-1}\mathfrak{p}$, say $(a/s)(a'/s') = x/t$ with $x \in \mathfrak{p}$ and $t \in S$. Then there exists $t^* \in S$ such that $t^*taa' = t^*ss'x$. Since the right side of the last equation lies in $\mathfrak{p}$, we get $t^*taa' \in \mathfrak{p}$. Now t^* and t do not lie in $\mathfrak{p}$, and since $\mathfrak{p}$ is prime, we conclude that either a or a' lies in $\mathfrak{p}$, whence at least one of a/s or a'/s' lies in $S^{-1}\mathfrak{p}$. □

6. Let $\mathfrak{P}$ be a prime ideal of $S^{-1}A$. Show that $\mathfrak{p} := i_A^{-1}(\mathfrak{P})$ is a prime ideal of A disjoint from S, and $S^{-1}\mathfrak{p} = \mathfrak{P}$. Here, $i_A\colon A \to S^{-1}A$ is the canonical map defined in (#). Conclude that there is a bijective correspondence between prime ideals of $S^{-1}A$ and prime ideals of A disjoint from S.

Solution: From the material done in class we know that $\mathfrak{p} := i_A^{-1}(\mathfrak{P})$ is a prime ideal in A since the inverse image of any prime ideal under a ring homomorphism is again a prime ideal. It remains to show that $S^{-1}\mathfrak{p} = \mathfrak{P}$. Suppose $x/s \in S^{-1}\mathfrak{p}$, with $x \in \mathfrak{p}$ and $s \in S$. By defintion of i_A and of $\mathfrak{p}$, we see that $x/1 \in \mathfrak{P}$, whence $x/s = (1/s)(x/1) \in \mathfrak{P}$. Thus $S^{-1}\mathfrak{p} \subset \mathfrak{P}$. Conversely, suppose $a/s \in \mathfrak{P}$. Then $a/1 = s(a/s)$ also lies in $\mathfrak{P}$, i.e. $a \in i_A^{-1}\mathfrak{P} = \mathfrak{p}$, proving that $a/s \in S^{-1}\mathfrak{p}$.

From Problem **5** and what we have established above, it is clear that $\mathfrak{p} \mapsto S^{-1}\mathfrak{p}$ and $\mathfrak{P} \mapsto i_A^{-1}\mathfrak{P}$ establishes the required bijective correspondence, with each map above being the inverse of the other. □

7. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of A with $\mathfrak{p} \supset \mathfrak{q}$. Let $S = A \smallsetminus \mathfrak{p}$.

- **(a)** Show that $S^{-1}\mathfrak{q}$ is a prime ideal in $A_\mathfrak{p}$ and that $S^{-1}\mathfrak{q} = \mathfrak{q}A_\mathfrak{p}$.
- **(b)** Show that $Q(A/\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$. Here $Q(A/\mathfrak{p})$ is the quotient field (i.e. the field of fractions) of the integral domain $A/\mathfrak{p}$.
- **(c)** Show that $A_\mathfrak{p}$ is a local ring with $\mathfrak{p}A_\mathfrak{p}$ its unique maximal ideal.

Solution: (a) Problem **5** shows that $S^{-1}\mathfrak{q}$ is a prime ideal of $A_\mathfrak{p}$. We claim that $\mathfrak{q}A_\mathfrak{p} = S^{-1}\mathfrak{q}$. Let $\theta \in \mathfrak{q}A_\mathfrak{p}$. Then $\theta = \sum_{i=1}^n q_i(a_i/s_i)$, where $q_i \in \mathfrak{q}$, $a_i \in \mathfrak{p}$, and $s_i \in S = A \smallsetminus \mathfrak{p}$. Let $s = \prod_I s_i$ and $t_i = s_1 s_1 \dots s_{i-1}s_{i+1}\dots s_n$, for $i = 1, \dots, n$ (with an obvious interpretation when i equals either 1 or n). Then

$$\theta = \frac{t_i q_i a_i}{s}.$$

It follows that $\theta \in S^{-1}\mathfrak{q}$. Conversely, if $q/s \in S^{-1}\mathfrak{q}$, then $q/s = q(1/s) \in \mathfrak{q}A_\mathfrak{p}$. □

(b) It is simpler to prove a more general result, namely, if $\phi\colon A \to B$ is a ring map, S a multiplicative system in A, T the multiplicative system $\phi(S)$ in B, and M a B-module, then $S^{-1}M = T^{-1}M$, where for the first localization, we regard M as an A-module in the obvious way. The idea is to identify m/s with $m/\phi(s)$. In greater detail, let $S^{-1}M \to T^1 M$ be the map $m/s \mapsto m/\phi(s)$. This is well-defined, for, if $m/s = 0$, then there exists $s' \in S$ such that $s'm = 0$, i.e. $\phi(s')m = 0$, whence $m/(1_B) = 0$, i.e. $m/\phi(s) = 0$. The map is clearly surjective. Finally, note that if $m/\phi(s) = 0$ then there exists $t \in T$ such that $tm = 0$. Now $t = \phi(s')$ for some $s' \in S$, whence $s'm = \phi(s')m = 0$. If follows that $m/(1_A) = 0$, i.e. $m/s = 0$. The isomorphism is canonical and functorial and hence we write $S^{-1}M = T^{-1}M$.

In our case, if we set $B = A/\mathfrak{p}$, $\phi\colon A \to B$ the natural map of A to its quotient $A/\mathfrak{p}$, and $M = A/\mathfrak{p}$, then the above considerations show that $S^{-1}(A/\mathfrak{p}) = (\phi(S))^{-1}(A/\mathfrak{p})$. Now $\phi(S)$ is precisely the set of non-zero elements of the integral domain $A/\mathfrak{p}$. Thus $S^{-1}(A/\mathfrak{p}) = Q(A/\mathfrak{p})$. On the other hand, by problem **2**, $S^{-1}(A/\mathfrak{p}) = (S^{-1}A)/(S^{-1}\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$, the last equality coming from part **(a)** above. □

(c) Since $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ is a field (in fact equal to $Q(A/\mathfrak{p})$), $\mathfrak{p}A_\mathfrak{p}$ is a maximal ideal of $A_\mathfrak{p}$. Suppose J is an ideal of $A_\mathfrak{p}$ and let $I = i_A^{-1}(J)$. If I contains an element of S, say s, then $s/1$ is a unit of $A_\mathfrak{p}$ in J, which means $J = A_\mathfrak{p}$. Thus, if J is a proper ideal of $A_\mathfrak{p}$, then $I \cap S\emptyset$. It is easy to see that $S^{-1}I = J$ by repeating the arguments given for the solutions of many of the problems above. Thus $I \subset \mathfrak{p}$. Since localizations preserve inclusions (see **(c)** of Problem **1**), we see that $J \subset S^{-1}\mathfrak{p} = \mathfrak{p}A_\mathfrak{p}$ (the last equality is from part **(a)**). Thus $\mathfrak{p}A_\mathfrak{p}$ is the only maximal ideal of $A_\mathfrak{p}$. □