HW 1

Due date: Jan 28, 2022

Throughout, if A is a ring[1] then Mod_A denotes the category of A-modules. If $M, N \in \mathrm{Mod}_A$, we will often use the phrase "$f\colon M \to N$ is an A-map" as a shorthand for "$f\colon M \to N$ is a homomorphism of A-modules".

For an element $a \in A$ and an A-module M, $\mu_a\colon M \to M$ will denote the A-map $x \mapsto ax$. The map μ_a is often called the "multiplication by a" map.

Localization as a functor. In what follows A is a ring, and $S \subset A$ a multiplicative system. If $M \in \mathrm{Mod}_A$, we write

$$i_M\colon M \to S^{-1}M \tag{\#}$$

for the localization map $m \mapsto m/1$.

For an A-map $f\colon M \to N$ we define a map

$$S^{-1}f\colon S^{-1}M \longrightarrow S^{-1}N$$

by the rule $m/s \mapsto f(m)/s$.

1. Show the following:
- **(a)** $S^{-1}f$ is well-defined.
- **(b)** If $g\colon L \to M$ is a second map, then $S^{-1}(f \circ g) = (S^{-1}f) \circ (S^{-1}g)$.
- **(c)** If f is injective, so is $S^{-1}f$.
- **(d)** If f is surjective, so is $S^{-1}f$.

2. Let L be an A-submodule of M. Regard $S^{-1}L$ as a submodule of $S^{-1}M$ via the part **(c)** of Problem **1**. Show that the map

$$S^{-1}(M/L) \longrightarrow (S^{-1}M)/(S^{-1}L)$$

given by $\frac{m+L}{s} \mapsto \frac{m}{s} + S^{-1}L$, is an isomorphism. (In what follows, and for the rest of the course, we will identify (without comment) $S^{-1}(M/L)$ with $(S^{-1}M)/(S^{-1}L)$ via the above isomorphism.)

3. Let $\ker(f)$, $\mathrm{im}(f)$, and $\mathrm{coker}(f)$ denote the kernel, image, and cokernel of f respectively. Show that
- **(a)** $S^{-1}\ker(f) = \ker(S^{-1}f)$, where we regard both sides as submodules of $S^{-1}M$;
- **(b)** $S^{-1}\mathrm{im}(f) = \mathrm{im}(S^{-1}f)$;
- **(c)** $S^{-1}\mathrm{coker}(f) = \mathrm{coker}(S^{-1}f)$.

[1] According to our conventions, commutative and with identity.

Localization of prime ideals. As before, A is a ring and $S \subset A$ a multiplicative system. Note that by part **(c)** of Problem **1**, if I is an ideal of A then $S^{-1}I$ can be regarded as an ideal of $S^{-1}A$. In what follows, we will so regard it.

If $f \colon A \to B$ is a ring homomorphism, and I an ideal in A, then IB will denote the ideal in B generated by the image of I in B. In other words, IB consists of finite sums of the form $\sum_\alpha b_\alpha f(x_\alpha)$, where the x_α lie in I and the b_α in B.

4. Let I be an ideal of A. Show that $S^{-1}I = S^{-1}A$ if and only if $S \cap I \neq \emptyset$.

5. Let $\mathfrak{p}$ be a prime ideal of A such that $S \cap \mathfrak{p} = \emptyset$. Show that $S^{-1}\mathfrak{p}$ is a prime ideal of $S^{-1}A$.

6. Let $\mathfrak{P}$ be a prime ideal of $S^{-1}A$. Show that $\mathfrak{p} := i_A^{-1}(\mathfrak{P})$ is a prime ideal of A disjoint from S, and $S^{-1}\mathfrak{p} = \mathfrak{P}$. Here, $i_A \colon A \to S^{-1}A$ is the canonical map defined in (#). Conclude that there is a bijective correspondence between prime ideals of $S^{-1}A$ and prime ideals of A disjoint from S.

7. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of A with $\mathfrak{p} \supset \mathfrak{q}$. Let $S = A \setminus \mathfrak{p}$.

- **(a)** Show that $S^{-1}\mathfrak{q}$ is a prime ideal in $A_\mathfrak{p}$ and that $S^{-1}\mathfrak{q} = \mathfrak{q}A_\mathfrak{p}$.
- **(b)** Show that $Q(A/\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$. Here $Q(A/\mathfrak{p})$ is the quotient field (i.e. the field of fractions) of the integral domain $A/\mathfrak{p}$.
- **(c)** Show that $A_\mathfrak{p}$ is a local ring with $\mathfrak{p}A_\mathfrak{p}$ its unique maximal ideal.