Feb 8, 2022 — Lecture 9 — Math 300

Recall that a polynomial $p(z)$ with complex coefficients and with $\deg p > 0$ can be written uniquely as

$$p(z) = c\,(z-w_1)^{e_1}(z-w_2)^{e_2}\cdots(z-w_m)^{e_m}$$

where c is a non-zero constant, $e_1, \dots, e_m$ are positive integers, and $w_1, \dots, w_m$ are the distinct roots of $p(z)$.

Recall that a rational function $f(z)$ is one which is the ratio of two polynomials; $f = P/q$, where p and q are polynomials and q is not the polynomial which is identically zero. Such a function can be written as

$$f(z) = c\,\frac{(z-w_1)^{e_1}\cdots(z-w_m)^{e_m}}{(z-z_1)^{d_1}\cdots(z-z_k)^{e_k}} \qquad (c \neq 0 \text{ constant}).$$

We may assume, after cancelling all common factors, that the w_j are distinct (i.e. if $i \neq j$, then $w_i \neq w_j$), the z_i are distinct, and no w_j equals any z_i.

If $f = P/q$ is s.t. $\deg q < \deg p$, then by the Euclidean algorithm for dividing polynomials we know that $p(z) = a(z)\,q(z) + h(z)$, with $\deg h < \deg q$, where $a(z)$ is the "quotient" when p is divided by q and h is the remainder. In this case

$$f(z) = a(z) + \frac{h(z)}{q(z)}$$

and the rational function h/q is s.t. $\deg q > \deg h$. Since we understand polynomials like $a(z)$, to study rational functions we need to understand rational functions in which the polynomial in the denominator has a greater degree than the polynomial in the numerator.

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In view of the discussion in the previous page, let

$$f(z) = c \, \frac{(z-w_1)^{e_1} (z-w_2)^{e_2} \cdots (z-w_m)^{e_m}}{(z-z_1)^{d_1} (z-z_2)^{d_2} \cdots (z-z_k)^{d_k}}$$

with

- $d_1 + \cdots + d_k > e_1 + \cdots + e_m$
- w_j's distinct; z_i's distinct, no w_j equal to any z_i.
- c a non-zero constant.

Theorem: In the above situation there exist unique complex numbers A_{ij} such that

$$f(z) = \frac{A_{10}}{(z-z_1)^{d_1}} + \frac{A_{11}}{(z-z_1)^{d_1-1}} + \cdots + \frac{A_{1,d_1-1}}{z-z_1}$$

$$+ \frac{A_{20}}{(z-z_2)^{d_2}} + \frac{A_{21}}{(z-z_2)^{d_2-1}} + \cdots + \frac{A_{2,d_2-1}}{z-z_2}$$

$$\vdots$$

$$+ \frac{A_{k0}}{(z-z_k)^{d_k}} + \frac{A_{k1}}{(z-z_k)^{d_k-1}} + \cdots + \frac{A_{k,d_k-1}}{z-z_k}$$

Remark: We will not be proving the above theorem. It is an algebraic statement, and not a complex analytic statement. However it is a very useful theorem and so we give an example of such a break-up. The above decomposition of the rational function f is called the partial fraction decomposition of f.

Example: Find the partial fraction decomposition of

$$f(z) = \frac{8z^2 - 17z + 7}{(z-1)^2 (z-2)}$$

Solution: The form of the partial fraction decomposition of f is:

$$f(z) = \frac{a}{(z-1)^2} + \frac{b}{z-1} + \frac{c}{z-2}. \qquad (*)$$

We have to find $a, b,$ and c. The usual technique is to "clear denominators" and write $8z^2 - 17z + 7$ as $a(z-2) + b(z-1)(z-2) + c(z-1)^2$, compare coefficients, and solve for $a, b,$ and c from the resulting equations. Here is a different technique.

Multiply both sides of $(*)$ by $(z-1)^2$. Get

$$(z-1)^2 f(z) = a + b(z-1) + \frac{c(z-1)^2}{z-2}$$

Hence $\lim\limits_{z\to 1} (z-1)^2 f(z) = a.$

So $$a = \lim_{z\to 1} (z-1)^2 f(z) = \lim_{z\to 1} \frac{8z^2 - 17z + 7}{z-2}$$

$$= -2/(-1) = 2.$$

Thus $a = 2$.

Next multiply both sides of $(*)$ by $z-2$. Get

$$(z-2) f(z) = \frac{a(z-2)}{(z-1)^2} + \frac{b(z-2)}{z-1} + c$$

Hence $\lim\limits_{z\to 2} (z-2) f(z) = c$. Thus

$$c = \lim_{z\to 2} (z-2) f(z) = \lim_{z\to 2} \frac{8z^2 - 17z + 7}{(z-1)^2}$$

$$= \frac{8(4) - 17(2) + 7}{1^2} = 5.$$

Thus $c = 5$.

Finally note that

$$\frac{d}{dz}\left\{(z-1)^2 f(z)\right\} = \frac{d}{dz}\left\{a + b(z-1) + \frac{c(z-1)^2}{z-2}\right\}$$

$$= b + \frac{(z-2)(2c(z-1)) - c(z-1)^2}{(z-2)^2}$$

$$\longrightarrow b \quad \text{as } z \to 1.$$

Thus $$b = \lim_{z \to 1} \frac{d}{dz} \left\{ (z-1)^2 f(z) \right\}$$

$$= \lim_{z \to 1} \frac{d}{dz} \left\{ \frac{8z^2 - 17z + 7}{z-2} \right\}$$

$$= \lim_{z \to 1} \left\{ \frac{(z-2)(16z-17) - (8z^2 - 17z + 7)}{(z-2)^2} \right\}$$

$$= \frac{(-1)(-1) - (-2)}{(-1)^2} = 3.$$

Thus $b = 3$.

In summary: $a = 2$, $b = 3$, $c = 5$. In other words

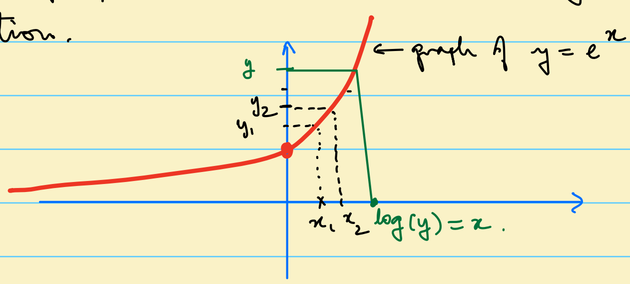
$$\frac{8z^2 - 17z + 7}{(z-1)^2 (z-2)} = \frac{2}{(z-1)^2} + \frac{3}{z-1} + \frac{5}{z-2}.$$

is the required partial fraction decomposition.

Exponentials and logarithms:

Recall that for $z = x + iy$, we defined
$$e^z = e^x (\cos(y) + i \sin(y)).$$

In the real variable case, e^x is an increasing function (see graph below) and hence $y = e^x$ defines a one-to-one function.



If $x_1 \neq x_2$ then $y_1 \neq y_2$.

Since $\exp(x) = e^x$ is a one-to-one function, it can be inverted and we have the inverse function ~~ln (y)~~ = $\log(y)$.

In the complex case is $f(z) = e^z$ one-to-one? We answer that question in the next page.

Important! We will no longer use the symbol $\ln(x)$ for the natural logarithm. From now on, this will be written as $\log(x)$.

Suppose $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers such that
$$e^{z_1} = e^{z_2}.$$
This means
$$\underbrace{e^{x_1}\left(\cos(y_1) + i\sin(y_1)\right)}_{\text{Polar decomposition of } e^{z_1}} = \underbrace{e^{x_2}\left(\cos(y_2) + i\sin(y_2)\right)}_{\text{Polar decomposition of } e^{z_2}.}$$

Since the "r" part of a polar decomposition is unique, this means
$$e^{x_1} = e^{x_2}.$$
Since e^x is one-to-one on $\mathbb{R}$, this means
$$x_1 = x_2.$$
Note $\quad e^{x_1} = |z_1|$ and $e^{x_2} = |z_2|$.

So $\quad x_1 = \log|z_1| = \log|z_2| = x_2.$

The above polar decomposition also gives
$$\cos(y_1) + i\sin(y_1) = \cos(y_2) + i\sin(y_2)$$
i.e.
$$\cos(y_1) = \cos(y_2) \quad \text{and} \quad \sin(y_1) = \sin(y_2).$$
This happens if and only if
$$y_1 = y_2 + 2\pi n \quad \text{for some integer } n.$$
In particular if $z_1 = x_1 + iy_1$ and $z_2 = x_1 + i(y_1 + 2\pi)$, then $z_1 \neq z_2$ but $e^{z_1} = e^{z_2}$.

$$\boxed{\textbf{Thus } f(z) = e^z \textbf{ is } \underline{\textbf{NOT}} \textbf{ one-to-one!}}$$

This creates problems for the definition of logarithms.

Let us look at the problem in a slightly different way. Consider the equation (for $z \neq 0$)
$$e^w = z$$
where we are asked to solve for w (with z being given).

Write $w = a + ib$. Then
$$z = e^a e^{ib}.$$
It follows that b is an argument of z. We write
$$b = \arg(z)$$
with the understanding that $\arg(z)$ is **multivalued**.
(We had earlier regarded $\arg(z)$ as a set, but we will use the symbol also for a member of the set.)

Moreover, $|z| = |e^a| \cdot |e^{ib}| = e^a$,
whence
$$a = \log |z|.$$
This gives the "formula"
$$\log(z) = \underbrace{\log |z|}_{\text{single valued}} + \arg(z).$$

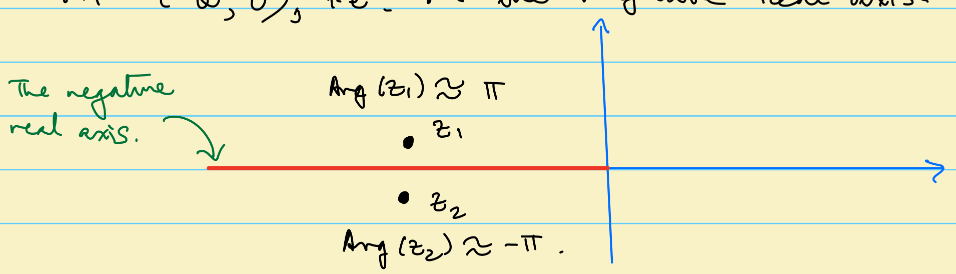
multivalued because $\arg(z)$ is multivalued

multivalued

Since $\arg(z)$ is multi-valued, so is $\log(z)$, i.e. log is <u>NOT</u> an honest function. For any $z \neq 0$, $\log(z)$ is really a set of values: $\log(z) = \{\log|z| + i\,\mathrm{Arg}(z) + 2\pi i n \mid n \in \mathbb{Z}\}$, where $\mathrm{Arg}(z)$ is the principal argument.

Recall that $\mathrm{Arg}(z)$ is the unique argument of z in the interval $(-\pi, \pi]$.

We also saw that $\mathrm{Arg}(z)$ is <u>not</u> continuous on $(-\infty, 0)$, i.e. on the negative real axis.



We define **the principal logarithm** to be the function

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Log on the domain $D = \mathbb{C} \setminus (-\infty, 0]$

$$\mathrm{Log} : D \longrightarrow \mathbb{C}$$

given by the formula

The principal argument.

The Principal logarithm

$$\mathrm{Log}(z) = \log|z| + i\,\mathrm{Arg}(z)$$

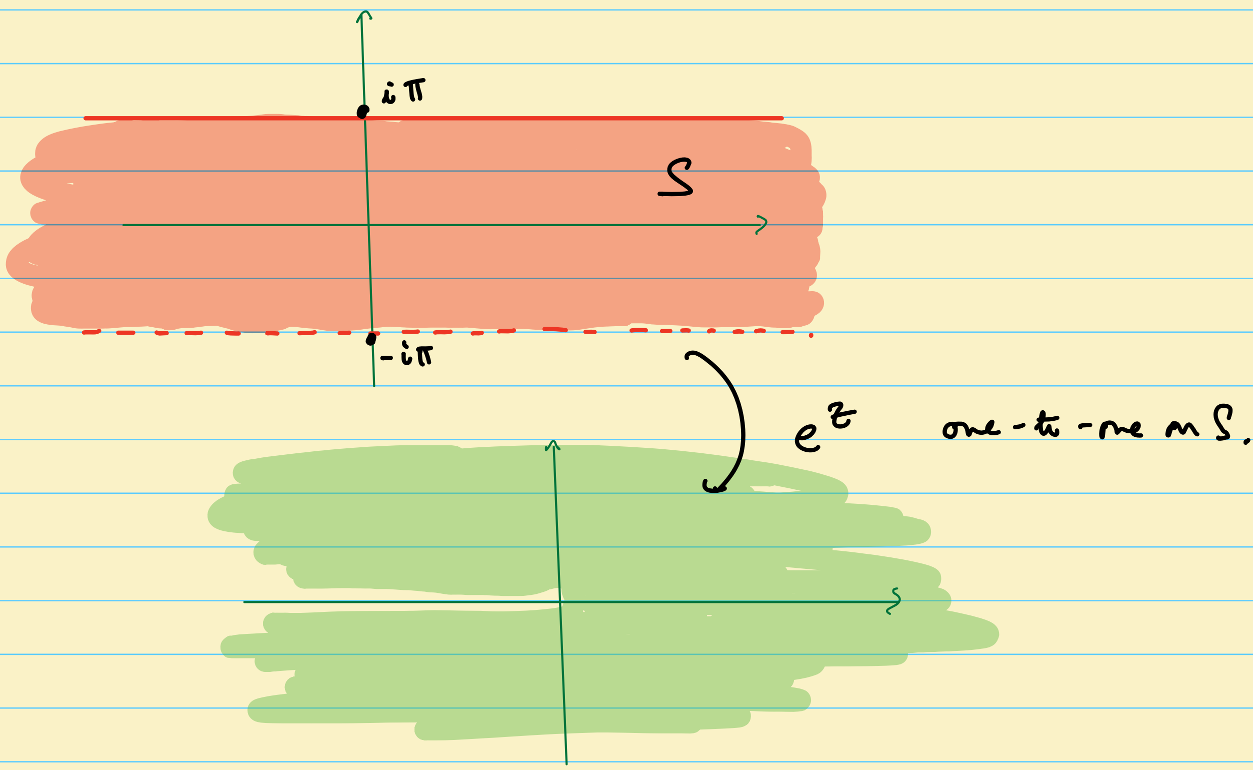
Since Arg is single-valued, so is Log.

Thus we have got ourselves an honest logarithm. At a small price. We had to remove the negative real axis from our considerations. Log is only defined on

$$D = \mathbb{C} \setminus (-\infty, 0).$$

The real reason this works is the following observation: Though $f(z) = e^z$ is not one-to-one, it is one-to-one when restricted to the strip

$$S = \{ z \in \mathbb{C} \mid -\pi < \mathrm{Im}(z) \leq \pi \}$$



The image of the open strip $\{z \mid -\pi < \text{Im}(z) < \pi\}$ under the exponential map is D. Which is why $f(z) = e^z$ has an inverse (i.e. a logarithm) on D.

We will show later (next lecture)

$$\text{Log} : D \longrightarrow \mathbb{C}$$

is analytic on D and

$$\frac{d}{dz} \text{Log} = \frac{1}{z}.$$

To prove analyticity, we need the polar form of the Cauchy-Riemann equations.

Let $f = u + iv$ on an open set G. The polar form of the Cauchy-Riemann equation are

Polar form of the CR equations

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}}$$

Here u and v are regarded a functions of r and θ via $u(r,\theta) = \text{Re}(f(re^{i\theta}))$ & $v(r,\theta) = \text{Im}(f(re^{i\theta}))$.

Analyticity of Log(z) using the polar form of CR

We have

$$\text{Log}(z) = \text{Log}(re^{i\theta}) = \log r + i\theta$$

where $r = |z|$ and $\theta = \text{Arg}(z)$. It follows that if $\text{Log} = u + iv$, then

$$u(r,\theta) = \log r \quad \text{and} \quad v(r,\theta) = \theta.$$

Thus

$$\frac{\partial u}{\partial r} = \frac{1}{r} \qquad \frac{\partial v}{\partial r} = 0$$

$$\frac{\partial u}{\partial \theta} = 0 \qquad \frac{\partial v}{\partial \theta} = 1$$

It is clear that Log satisfies (the polar form of) the Cauchy-Riemann equs & hence is analytic.

Next lecture

- Will prove polar form of C-R
- Will show

$$\frac{d}{dz} \operatorname{Log}(z) = \frac{1}{z}$$

- Will define $\cos(z)$, $\sin(z)$
- Will define hyperbolic functions $\cosh(z)$, $\sinh(z)$.
- Perhaps start on CONTOUR INTEGRATION.