The Laplace equation:

Let u be a real-valued function on an open set G of $\mathbb{C}$ such that $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ exist on G. The Laplace equation for u on G is:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Definition: Let G be a domain in $\mathbb{C}$. A function $u: G \longrightarrow \mathbb{C}$ is said to be harmonic if all its second partial derivatives $\left(\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y \partial x}, \frac{\partial^2 u}{\partial y^2}\right)$ exist, are continuous, and u satisfies the Laplace equation.

Where do harmonic functions occur?

Let f be analytic on an open set G, say $f = u + iv$, then one can show that all the 2nd partials of u exist.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) \quad \text{(by Cauchy-Riemann)}$$

$$= \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 v}{\partial y \partial x}$$

$$= \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 v}{\partial x \partial y}$$

$$= \frac{\partial^2 u}{\partial x^2} - \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)$$

$$= \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x^2} \quad \left(\text{since } \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \text{ by C-R}\right)$$

$= 0.$

Conclusion: u is harmonic.

Since $g = -if$ is analytic, and $v = \mathrm{Re}(g)$, therefore v is also harmonic.

Theorem: Let $f = u + iv$ be analytic on a domain G, then u and v are harmonic.

Definition: Let u be harmonic on an open set G. A harmonic function v on G is said to be conjugate harmonic to u if $u + iv$ is analytic on G.

Suppose v_1, v_2 are two harmonic functions conjugate to u. Then

$$f_1 = u + iv_1 \qquad \text{and} \quad f_2 = u + iv_2$$

are both analytic on G. Assume G is a domain. Now $f_1 - f_2 = i(v_1 - v_2)$ is analytic on G and purely imaginary-valued. In the last lecture we showed that any analytic function which is purely imaginary valued is a constant. So $\exists\ c \in \mathbb{R}$ such that

$$f_1(z) - f_2(z) = ic \qquad \forall\ z \in G.$$

Hence $$i(v_1 - v_2) = ic$$

i.e.

$$v_1 - v_2 = c.$$

Conclusion: Two conjugates of a harmonic function u differ by a constant.

Examples:

1. Let $u(x,y) = e^x \cos y$. $G = \mathbb{C}$.

$$\frac{\partial^2 u}{\partial x^2} = e^x \cos y, \qquad \frac{\partial^2 u}{\partial y^2} = -e^x \cos y.$$

Hence $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ on $\mathbb{C}$

So u is harmonic.

Let us find a conjugate v for u.

Know (from CR)

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \underline{\text{AND}} \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

So

$$\frac{\partial v}{\partial x} = e^x \sin y \quad \text{AND} \quad \frac{\partial v}{\partial y} = e^x \cos y.$$

Now $\frac{\partial v}{\partial x} = e^x \sin y$

$$\Rightarrow \quad v(x,y) = \int e^x \sin y \, dx = e^x \sin y + \phi(y)$$

for some function ϕ.

Use this in the eqn $\frac{\partial v}{\partial y} = e^x \cos y$.

Get

$$e^x \cos y + \phi'(y) = e^x \cos y.$$

$$\Rightarrow \quad \phi'(y) = 0 \Rightarrow \phi = \text{const} = c \text{ (say)}.$$

Hence $v(x,y) = e^x \sin y + c$.

Note $u + iv = e^x \cos y + i e^x \sin y + ic$

$$= e^z + ic.$$

2. Show that

$$u(x,y) = x^3 - 3xy^2$$

is harmonic, and find a conjugate v of u.

<u>Soln</u>:

$$\frac{\partial^2 u}{\partial x^2} = 6x \quad , \quad \frac{\partial^2 u}{\partial y^2} = -6x$$

So $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, i.e. u is harmonic.

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \quad , \quad \frac{\partial u}{\partial y} = -6xy.$$

This means

$$\frac{\partial v}{\partial y} = 3x^2 - 3y^2 \quad , \quad \frac{\partial v}{\partial x} = 6xy.$$

The first of these relations gives

$$v = 3x^2y - y^3 + \phi(x).$$

This gives

$$\frac{\partial v}{\partial x} = 6xy + \phi'(x)$$

Hence $6xy = 6xy + \phi'(x)$.

In other words $\phi'(x) = 0$, i.e. $\phi(x) = c$.

Hence

$$v(x,y) = 3x^2y - y^3 + c.$$

What is $f = u + iv$?

Write $z = x + iy$.

$$f(z) = u(x,y) + iv(x,y)$$
$$= x^3 - 3xy^2 + i(3x^2y - y^3) + ic.$$

$$= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 + ic$$
$$= (x+iy)^3 = z^3 + ic.$$

So $f(z) = z^3 + ic, \quad c \in \mathbb{R}$.

An out of turn theorem (should have done it last class):

Theorem: Let f be analytic on a domain G in $\mathbb{C}$. If $f'(z) \equiv 0$ on G, then $f(z)$ is a constant.

Proof:

Write $z = x+iy$, $f = u + iv$.

We know $f'(z) = \frac{\partial u}{\partial x}(x,y) + i \frac{\partial v}{\partial x}(x,y)$

This means (since $f' \equiv 0$) that

$$\left.\begin{array}{l} \frac{\partial u}{\partial x} \equiv 0 \quad \text{and} \quad \frac{\partial v}{\partial x} \equiv 0 \\ \text{But } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \text{ Hence } \frac{\partial u}{\partial y} \equiv 0. \end{array}\right\} \text{ on } G$$

Thus $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} \equiv 0.$ on G

Similarly $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} \equiv 0$

Since G is connected, this means u and v are constants, and hence f is a constant. //

Polynomial and Rational functions (again)

Let p be a polynomial of degree n, and $z_0 \in \mathbb{C}$.

$$p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$$

Can write $p(z)$ in the following way:-

$$p(z) = b_0 + b_1(z-z_0) + b_2(z-z_0)^2 + \dots + b_n(z-z_0)^n.$$

To see this, write
$$p(z) = p\big((z-z_0)+z_0\big)$$
and then use binomial thm.

Another way.
$$p(z) = a_0 + a_1 z + \dots + a_n z^n$$
$$p(z) = b_0 + b_1(z-z_0) + \dots + b_n (z-z_0)^n.$$

Let us work out the b_i's.

Have
$$p(z_0) = b_0$$
Next,
$$p'(z) = b_1 + 2b_2(z-z_0) + \dots + n b_n (z-z_0)^{n-1}.$$
So
$$\left.\begin{aligned} p'(z_0) &= b_1 \\ p''(z_0) &= 2b_2 \\ &\vdots \\ p^{(k)}(z_0) &= k!\, b_k \\ &\vdots \\ p^{(n)}(z_0) &= n!\, b_n \end{aligned}\right\} \Rightarrow \quad b_k = \frac{p^{k}(z_0)}{k!}$$

So can write
$$p(z) = b_0 + b_1(z-z_0) + \dots + b_n(z-z_0).$$

Suppose z_0 is a root of $p(z)$, and $\deg p \geq 1$. Then $p(z_0)=0$, i.e. $b_0 = 0$, since $b_0 = p(z_0)$. This means
$$p(z) = b_1(z-z_0) + b_2(z-z_0)^2 + \dots + b_n(z-z_0)^n$$
$$= (z-z_0)\, q(z), \quad q(z) = b_1 + b_2(z-z_0) + \dots + b_n(z-z_0)^{n-1}$$

We have proved the following:

Proposition: Let $p(z)$ be a polynomial of degree $n \geq 1$. Then

(a) z_0 is a root of $p(z)$ if and only if
$p(z) = (z - z_0)\, q(z)$ where q is a polynomial of degree $n-1$.

(b) $p(z)$ has at most n roots.

Proof: Part (a) follows from our calculations in the previous page. Part (b) follows by applying (a) repeatedly.

Theorem (The Fundamental Theorem of Algebra) (Gauss): Let $p(z)$ be a polynomial with complex coefficients of degree ≥ 1. Then $p(z)$ has a root in $\mathbb{C}$.

Proof: Omitted. Perhaps (?) later in the course.

Corollary: Let $p(z)$ be a polynomial of deg ≥ 1, with complex coefficients. Then

$$p(z) = c(z - z_1)(z - z_2) \cdots (z - z_n)$$

where c is a non-zero constant and $z_1, \ldots, z_n$ are the roots of p.

Proof: By the fund'l thm, $p(z) = (z - z_1)\, q(z)$ where q is of deg $n-1$. Repeat this process until q's degree drops to zero. //

Next week: Suppose $R(z) = \dfrac{p(z)}{q(z)}$, $\deg q(z) > \deg p(z)$.

Write $q(z) = c\,(z - z_1)^{d_1} \cdots (z - z_k)^{d_k}$, $d_i \geq 1$, $z_1, \ldots, z_k$ distinct. Then

$\longrightarrow$ OVER

$$
\left.
\begin{aligned}
R(z) = {} & \frac{A_{10}}{(z-z_1)^{d_1}} + \frac{A_{11}}{(z-z_1)^{d_1-1}} + \cdots + \frac{A_{1,d_1-1}}{(z-z_1)} \\
+ {} & \frac{A_{20}}{(z-z_2)^{d_2}} + \frac{A_{21}}{(z-z_2)^{d_2-1}} + \cdots + \frac{A_{2,d_2-1}}{(z-z_2)} \\
+ {} & \qquad\qquad \vdots \\
+ {} & \frac{A_{k0}}{(z-z_k)^{d_k}} + \frac{A_{k1}}{(z-z_k)^{d_k-1}} + \cdots + \frac{A_{k,d_k-1}}{(z-z_k)}
\end{aligned}
\right\}
$$

The partial fraction decomposition of $R(z)$.