**Announcement:**

You do not have to submit the following problems for HW3:
2.2.5, 2.3.14, 2.3.15, 2.3.3.

**Cauchy-Riemann:**

$f = u + iv$ ; $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial x}$ exist at $(x_0, y_0)$ in domain of $f$

The CR-equations at $(x_0, y_0)$ are

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0)$$

and

$$\frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_1)$$

CR-eqns

$f$ may or may not satisfy the above equations.

We have shown:

Necessity of CR.

If $f$ is diff'ble at $z_0 = x_0 + iy_0$ then $f$ satisfies the CR equations at $z_0$.

We will now show "sufficiency" but with extra hypotheses on $f$.

**Theorem:** Let $G$ be open in $\mathbb{C}$, $z_0 = x_0 + iy_0$ a point in $G$ and $f : G \longrightarrow \mathbb{C}$ a function with $f = u + iv$ such that $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist ~~and are continuous~~ in a neighborhood of $z_0$ and are continuous at $z_0$

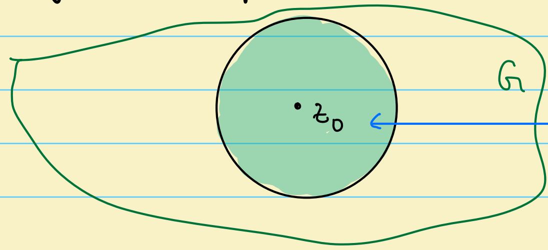
NOTE THE EDIT

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0)$$

$$\frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0)$$

then $f$ is diff'ble at $z_0$.

NOTE THE EDIT.




Need the partials to exist in a nbhd of $z_0$ and be continuous at $z_0$.

**Proof:**

Let $\Delta z = \Delta x + i\Delta y$. Then

$(x_0+\Delta x, y_0+\Delta y) = z_0 + \Delta z$

$$u(x_0+\Delta x, y_0+\Delta y) - u(x_0,y_0)$$
$$= \left[u(x_0+\Delta x, y_0+\Delta y) - u(x_0, y_0+\Delta y)\right] + \left[u(x_0, y_0+\Delta y) - u(x_0,y_0)\right]$$

By the Mean Value Theorem from Calculus the 1st bracketed expression

$$= \Delta x \frac{\partial u}{\partial x}(x^*, y_0+\Delta y), \quad \text{where } x^* \text{ lies between } x_0 \text{ and } x_0+\Delta x$$

$$= \Delta x \frac{\partial u}{\partial x}(x_0,y_0) + \varepsilon_1 \quad \text{where } \varepsilon_1 \to 0 \text{ as } x^* \to x_0$$

i.e. as $\Delta x \to 0$, i.e. as $\Delta z \to 0$

Since $\frac{\partial u}{\partial x}$ is continuous at $(x_0,y_0)$

The same reasoning shows that the 2nd bracketed expression

$$= \Delta y \frac{\partial u}{\partial y}(x_0,y_0) + \varepsilon_2, \quad \varepsilon_2 \to 0 \text{ as } \Delta z \to 0.$$

Similarly

$$v(x_0+\Delta x, y_0+\Delta y) - v(x_0, y_0+\Delta y) = \Delta x \frac{\partial v}{\partial x}(x_0,y_0) + \varepsilon_3$$
$$v(x_0, y_0+\Delta y) - v(x_0,y_0) = \Delta y \frac{\partial v}{\partial y}(x_0,y_0) + \varepsilon_4$$

$\varepsilon_3 \to 0$ & $\varepsilon_4 \to 0$ as $\Delta z \to 0$.

So $$v(x_0+\Delta x, y_0+\Delta y) - v(x_0,y_0) = \Delta x \frac{\partial v}{\partial x}(x_0,y_0) + \Delta y \frac{\partial v}{\partial y}(x_0,y_0) + \varepsilon_3 + \varepsilon_4$$

$$f(z_0+\Delta z) - f(z) = u(x_0+\Delta x, y_0+\Delta y) - u(x_0,y_0) + i\{v(x_0+\Delta x, y_0+\Delta y) - v(x_0,y_0)\}$$

$$f(z_0+\Delta z) - f(z_0) = \Delta x\left\{\frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0) + \varepsilon_1 + i\varepsilon_3\right\}$$
$$+ \Delta y\left\{\frac{\partial u}{\partial y}(x_0,y_0) + i\frac{\partial v}{\partial y}(x_0,y_0) + \varepsilon_2 + i\varepsilon_4\right\}$$

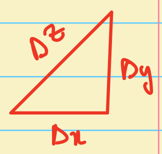
$$= (\Delta x + i\Delta y)\left(\frac{\partial u}{\partial x}(x_0,y_0)\right) + i(\Delta x + i\Delta y)\left(\frac{\partial v}{\partial x}(x_0,y_0)\right)$$
$$+ \varepsilon_1\Delta x + \varepsilon_2\Delta y + i(\varepsilon_3\Delta x + \varepsilon_4\Delta y)$$

Using the Cauchy-Riemann equations.

$$= \Delta z \left( \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0) \right) + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$
$$+ i(\varepsilon_3 \Delta x + \varepsilon_4 \Delta y)$$

Hence

$$(*) \quad \begin{cases} \dfrac{f(z_0+\Delta z) - f(z)}{\Delta z} = \dfrac{\partial u}{\partial x}(x_0, y_0) + i \dfrac{\partial v}{\partial x}(x_0, y_0) \\ \qquad + \dfrac{\varepsilon_1 \Delta x + \varepsilon_2 \Delta y}{\Delta z} + \dfrac{i(\varepsilon_3 \Delta x + \varepsilon_4 \Delta y)}{\Delta z}. \end{cases}$$




Note $|\Delta x| \le |\Delta z|$, $|\Delta y| \le |\Delta z|$.

So $$\left| \frac{\varepsilon_1 \Delta x + \varepsilon_2 \Delta y}{\Delta z} \right| \le \frac{\varepsilon_1 |\Delta x| + \varepsilon_2 |\Delta y|}{|\Delta z|} \le \varepsilon_1 + \varepsilon_2.$$

Similarly

$$\left| \frac{\varepsilon_3 \Delta x + \varepsilon_4 \Delta y}{\Delta z} \right| \le \varepsilon_3 + \varepsilon_4.$$

In other words

$$\lim_{\Delta z \to 0} \frac{\varepsilon_1 \Delta x + \varepsilon_2 \Delta y}{\Delta z} = 0$$

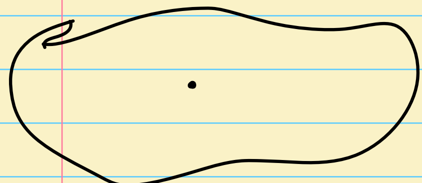
$$\lim_{\Delta z \to 0} \frac{\varepsilon_3 \Delta x + \varepsilon_4 \Delta y}{\Delta z} = 0$$

Now let $\Delta z \to 0$ in equation $(*)$. Get

$$\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0).$$

Since the limit on the left exists, $f$ is diff'ble at $z_0$. //

Reminder (The difference between analyticity and differentiability).



We say $f$ is analytic in an open set if it is diff'ble at every pt in that open set. We say $f$ is analytic at a point $z_0$ if $f$ is diff'ble at every point in a neighbourhood of $z_0$. So being diff'ble at a point $z_0$ does not ensure being analytic at $z_0$.

Example from last time: $f(z) = |z|^2$ is diff'ble at $0$ and nowhere else. So $f$ is diff'ble at $0$ but not analytic at $0$.

Recall that last time we used the above theorem to prove that $f(z) = e^z$ is analytic on $\mathbb{C}$. Indeed in this case, $u(x,y) = e^x \cos y$, $v(x,y) = e^x \sin y$. Clearly $u, v$ have partial derivatives which are cts, and it is easy to check (as we did last lecture) that they satisfy the CR-equations at all points.

We know:
$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$
$$= e^x \cos y + i e^x \sin y$$
$$= e^z.$$

Examples:

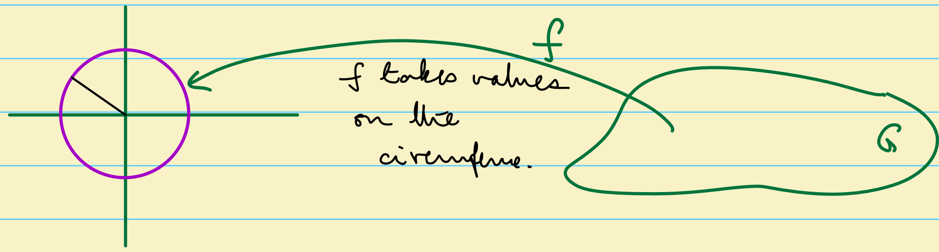
1. Suppose $f: G \longrightarrow \mathbb{C}$ is real-valued ($G$ a domain in $\mathbb{C}$) and analytic. Then we claim $f$ is a constant.

Have $f = u + iv$, with $v \equiv 0$. Hence $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$.
By CR this means $\frac{\partial u}{\partial x} \equiv 0$ and $\frac{\partial u}{\partial y} \equiv 0$.

Fact from several variables calculus: If $g: D \to \mathbb{R}$ is a function on a connected open set $D$ of $\mathbb{R}^2$ such that $\frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} \equiv 0$, then $g$ is a constant.

2. Suppose $G$ is a domain and $f: G \to \mathbb{C}$ is analytic and is purely imaginary. Then $f$ is a constant. Same proof as above.

3. Suppose $G$ is a domain and $f: G \to \mathbb{C}$ is an analytic function such that $|f|$ is constant. Then $f$ is constant.
Left as an exercise for you.
**Hint** : Use the fact that $u^2+v^2 =$ constant.




## Harmonic Functions

Let $G$ be an open set in $\mathbb{C}$ and $u: G \to \mathbb{R}$ a real function such that $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ exist. Then $u$ is said to be harmonic on $G$ if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

on $G$.

Exercise : Suppose $f = u + iv$ is analytic on $G$. Show that $u$ and $v$ are harmonic.