**Recall:** Suppose $f$ is a complex-valued function defined in a neighborhood of $z_0 \in \mathbb{C}$. We say $f$ is continuous at $z_0$ if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

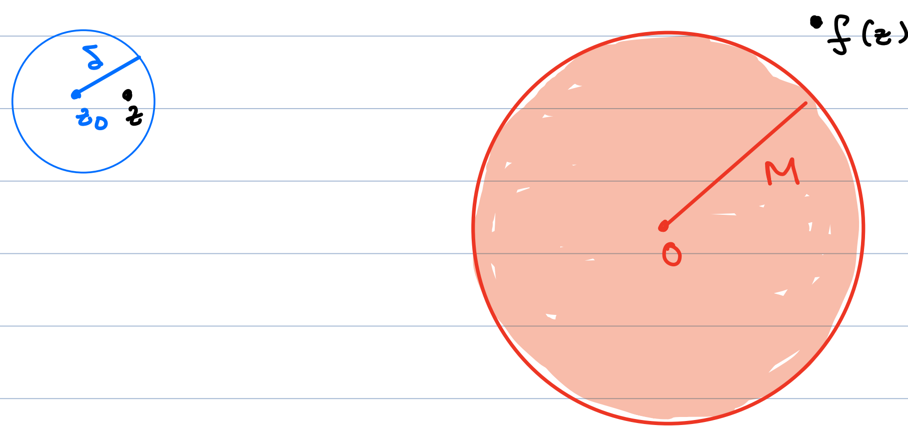
## Limits and infinity:

1. We say $\lim_{z \to z_0} f(z) = \infty$ if $\lim_{z \to z_0} |f(z)| = \infty$.

   According to real variable Calculus this means the following: Given a positive real number $M > 0$, there exists $\delta > 0$ such that

   $$|f(z)| > M$$

   whenever $0 < |z - z_0| < \delta$.




2. We also spoke about $\lim_{z \to \infty} f(z)$.

   Recall we defined it as

   $$L = \lim_{w \to 0} f\left(\tfrac{1}{w}\right).$$

This means, given $\varepsilon > 0$, $\exists\ \delta > 0$ such that whenever $0 < |w| < \delta$ we have

$$\left| f\left(\tfrac{1}{w}\right) - L \right| < \varepsilon.$$

Let $R = \frac{1}{\delta}$. Note that the above is equivalent to saying that whenever $|z| > R$, we have

$$| f(z) - L | < \varepsilon.$$

This gives us an alternative definition of $\lim_{z \to \infty} f(z)$.

**Definition (alternative defn):** Let $G$ be an unbounded set, and $f$ a function on $G$. We say

$$\lim_{z \to \infty} f(z) = L$$

if for every $\varepsilon > 0$ there exists an $R > 0$ such that

$$|f(z) - L| < \varepsilon$$

whenever $|z| > R$.

**Analyticity:** Recall that if $f$ is defined in a nbhd of a cplx number $z_0$, we say $f$ is diff'ble at $z_0$ if the limit

neighbourhood

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$\dfrac{f(z_0 + \Delta z) - f(z_0)}{z_0}$ is called the difference quotient at $z_0$

exists. In this case we call the above limit, the derivative of $f$ at $z_0$, and denote the limit by the symbols $f'(z_0)$ or $\frac{df}{dz}(z_0)$.

**Example:** Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the function $f(z) = \overline{z}$.
(Recall: if $z = a + ib$, then $\overline{z} := a - ib$.)
Then the difference quotient (for $\Delta z \neq 0$)

$$(*) \quad \left\{ \begin{aligned} \frac{f(z+\Delta z) - f(z)}{\Delta z} &= \frac{\overline{(z+\Delta z)} - \overline{z}}{\Delta z} \\ &= \frac{\overline{z} + \overline{\Delta z} - \overline{z}}{\Delta z} \\ &= \frac{\overline{\Delta z}}{\Delta z} \end{aligned} \right.$$

Write $\Delta z = \Delta x + i \Delta y$, with $\Delta x, \Delta y$ real.
Let $\Delta z \longrightarrow 0$ along the real-axis. Then $\Delta y = 0$ and $\Delta z = \Delta x$. Now $\overline{\Delta x} = \Delta x$, since $\Delta x$ is real, and $\overline{\Delta z} = \Delta z$. This means the computation in $(*)$ yields

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = 1. \qquad \text{———} \; (+)$$

Let $\Delta z \longrightarrow 0$ along the imaginary axis. Then $\Delta z = i \Delta y$. Now $\overline{(i\Delta y)} = \overline{(0 + i\Delta y)} = 0 - i\Delta y$
$= -i\Delta y$.

So $\overline{\Delta z} = -\Delta z$ in this case. This means the computations in $(*)$ yields:

$$\frac{f(z+\Delta z)-f(z)}{\Delta z} = -1. \qquad \text{——————} \qquad (\dagger\dagger)$$

Comparing $(\dagger)$ and $(\dagger\dagger)$ we see that

$\lim\limits_{\Delta z\to 0} \dfrac{f(z+\Delta z)-f(z)}{\Delta z}$ does not exist for any $z$.

<u>Conclusion:</u> The function $f(z)=\bar{z}$ is not differentiable at any $z\in\mathbb{C}$.

<u>Example:</u> Let $f(z)=|z|^2$.

<u>Claim</u>: This is di -able at $z=0$ and is not differentiable at any other $z$.

<u>Differentiability at $z=0$</u>:

$$\frac{f(0+\Delta z)-f(0)}{\Delta z} = \frac{|0+\Delta z|^2-|0|^2}{\Delta z}$$

$$= \frac{|\Delta z|^2}{\Delta z}$$

$$= \frac{(\Delta z)\,\overline{(\Delta z)}}{\Delta z}$$

$$= \overline{\Delta z} \longrightarrow 0 \quad \text{as} \quad \Delta z\to 0$$

(Reason: $\Delta z=\Delta x+i\Delta y$, $\overline{\Delta z}=\Delta x-i\Delta y$)

<u>Conclusion</u>: $z\longmapsto |z|^2$ is diff'ble at $0$

<u>Non-differentiability at $z\neq 0$</u>:

$f(z)= z\cdot\bar{z}$.

We know that $p$ and $q$ are diff'ble at $z_0$, then $p/q$ is diff'ble at $z_0$ (the theorem was stated in the last class) provided $q(z_0) \neq 0$.

Suppose $z \neq 0$. Then

$$\bar{z} = \frac{f(z)}{z}.$$

The function $z$ is diff'ble everywhere. If $f$ was diff'ble at $z$ $(z \neq 0)$, then $\bar{z}$ would be diff'ble at $z$. However we know (and have proved) that $\bar{z}$ is NOT diff'ble anywhere. So $f$ cannot be diff'ble at $z \neq 0$.

<u>Conclusion</u>: $|z|^2$ is differentiable at $z=0$, and <u>NOWHERE ELSE</u>!

<u>What are the functions we know that are diff'ble?</u>

Polynomials : $f(z) = a_0 + a_1 z + \dots + a_n z^n$.

Rational functions (on their domains of definition) : $R(z) = \dfrac{a_0 + a_1 z + \dots + a_n z^n}{b_0 + b_1 z + \dots + b_m z^m}$

with at least one $b_i$ non-zero.

$$\left( \text{Reminder: } \lim_{z \to \infty} R(z) = \begin{cases} \frac{a_n}{b_m} & \text{if } n = m \\ 0 & \text{if } m > n \\ \infty & \text{if } m < n. \end{cases} \right)$$

<u>Definitions</u>: 1. Let $f : G \to \mathbb{C}$ be a function with $G$ open. Then $f$ is said to be <u>analytic on $G$</u>, if it is differentiable at every point of $G$.

2. Let $z_0$ be an interior point of a set $S$ and $f: S \longrightarrow \mathbb{C}$ a function. We say $f$ is analytic at $z_0$ if there is a neighbourhood of $z_0$ on which $f$ is analytic.

(***) !!! The function $f(z) = |z|^2$ is diff'ble at $z_0 = 0$ but it is not analytic at $z_0 = 0$.

## The Cauchy-Riemann equations:

Let $z_0 \in \mathbb{C}$ and suppose $f$ is function defined in a nbhd of $z_0$ and suppose $f$ is differentiable at $z_0$.

Write:
$$\left.\begin{aligned} z_0 &= x_0 + i y_0 \\ z &= x + i y \\ f(z) &= u(z) + i v(z) \\ &= u(x,y) + i v(x,y) \end{aligned}\right\} \quad \begin{aligned} &x_0, y_0 \in \mathbb{R} \\ &x, y \in \mathbb{R} \\ &u(z) = u(x,y) \in \mathbb{R} \\ &v(z) = v(x,y) \in \mathbb{R}. \end{aligned}$$

Know $f'(z_0)$ exists since $f$ is diff'ble at $z_0$. This means
$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \quad \text{exists.}$$

Write $\quad \Delta z = \Delta x + i \Delta y.$

First let $\Delta z \to 0$ along the real axis. Then $\Delta z = \Delta x$, and the difference quotient
$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{1}{\Delta x}\left\{ \begin{aligned} &u(x_0 + \Delta x, y_0) + i v(x_0 + \Delta x, y_0) \\ &- u(x_0, y_0) - i v(x_0, y_0) \end{aligned} \right\}$$

Since the limit of the LHS exists as $\Delta z = \Delta x \to 0$, the limit of the RHS also exists. This means

$$\lim_{\Delta x \to 0} \frac{u(x_0+\Delta x, y_0) - u(x_0, y_0)}{\Delta x}$$

and

$$\lim_{\Delta x \to 0} \frac{v(x_0+\Delta x, y_0) - v(x_0, y_0)}{\Delta x}$$

exist i.e., $\frac{\partial u}{\partial x}(x_0, y_0)$ exists & $\frac{\partial v}{\partial x}(x_0, y_0)$ exists.

Moreover,

$$\boxed{f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)}$$

<u>Next</u>, let $\Delta z \to 0$ through the imaginary axis. Have

$$\Delta z = i\Delta y.$$

Now

$$\frac{f(z_0+\Delta z) - f(z)}{\Delta z} = \frac{1}{i\Delta y}\left\{ u(x_0, y_0+\Delta y) - u(x_0, y_0) + i\left[v(x_0, y_0+\Delta y) - v(x_0, y_0)\right]\right\}$$

$$= \frac{u(x_0, y_0+\Delta y) - u(x_0, y_0)}{i\Delta y} + i\,\frac{v(x_0, y_0+\Delta y) - v(x_0, y_0)}{i\Delta y}$$

Let $\Delta z = i\Delta y \to 0$. Get

$$f'(z_0) = \frac{1}{i}\frac{\partial u}{\partial y}(x_0, y_0) + i \cdot \frac{1}{i}\frac{\partial v}{\partial y}(x_0, y_0).$$

$$\boxed{f'(z_0) = \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0).}$$

From the two boxed equations we conclude that:

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0)$$

$$\frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0)$$

The Cauchy-Riemann eqns when $f$ is diff'ble at $z_0$.

**Example:** Let us revisit $f(z) = \bar{z}$.

Write $f = u + iv$, $z = x + iy$ etc.

$f(x, y) = x - iy$

$u(x,y) = x$, $v(x,y) = -y$.

$\frac{\partial u}{\partial x} = 1$, $\frac{\partial v}{\partial y} = -1$

$\Rightarrow \frac{\partial u}{\partial x}(x,y) \neq \frac{\partial v}{\partial y}(x,y)$ for any $z = x + iy$.

In other words $f$ is nowhere diff'ble on $\mathbb{C}$.

**Theorem:** Let $G$ be an open set, $f = u + iv$ a complex valued function on $G$ such that $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist on $G$ and are continuous on $G$. Suppose

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

on $G$. Then $f$ is analytic on $G$.

**Example:** Define $\exp : \mathbb{C} \longrightarrow \mathbb{C}$ by

$$\exp(z) = e^{\mathrm{Re}(z)}\{\cos(\mathrm{Im}(z)) + i\sin(\mathrm{Im}(z))\}.$$

Then $\exp(z)$ is analytic on $\mathbb{C}$.

Pf: Write $e^z$ for $\exp(z)$, and $z = x+iy$.

$f(z) = e^z = e^x(\cos y + i\sin y)$.

$$u(x,y) = e^x \cos y \quad , \quad v(x,y) = e^x \sin(y)$$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= e^x \cos y \;, & \frac{\partial v}{\partial y} &= e^x \cos(y) \\ \frac{\partial u}{\partial y} &= -e^x \sin y \;, & \frac{\partial v}{\partial x} &= e^x \sin y. \end{aligned}\right\} \begin{array}{l} \text{Continuous} \\ \text{on } \mathbb{R}^2 \\ = \mathbb{C} \end{array}$$

Clearly the Cauchy-Riemann equations hold.

So if you believe the theorem, $e^z$ is analytic.