Recall from last time:

Let z_0 be a complex number and f a complex valued function defined in a neighbourhood of z_0, (perhaps with $z_0 \notin \mathrm{Dom}(f)$)

$$z_0 = x_0 + i y_0 \qquad x_0, y_0 \in \mathbb{R}$$

$$f(z) = u(z) + i v(z), \qquad u(z), v(z) \in \mathbb{R}$$

$$= u(x,y) + i\, v(x,y)$$

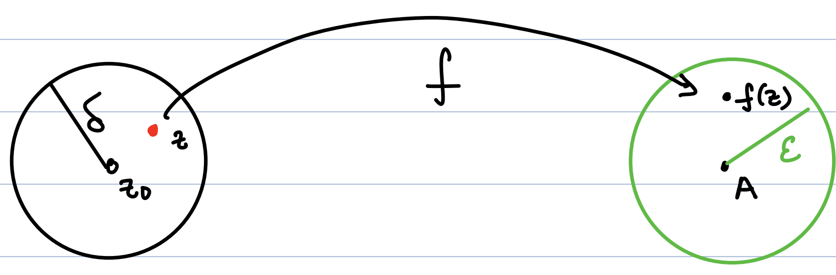
where $z = x + iy$, $x, y \in \mathbb{R}$.

We say

$$\lim_{z \to z_0} f(z) = A$$

or $f(z) \longrightarrow A$ as $z \to z_0$, if for every positive $\varepsilon > 0$, there exists $\delta > 0$ s.t. whenever $0 < |z - z_0| < \delta$, we have

$$|f(z) - A| < \varepsilon.$$



Because
$$|x - x_0| \le |z - z_0|$$
$$|y - y_0| \le |z - z_0|$$

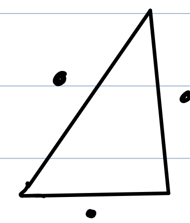
etc, one can easily see that

$$\lim_{z\to z_0} f(z) = \lim_{(x,y)\to(x_0,y_0)} u(x,y) + i \lim_{(x,y)\to(x_0,y_0)} v(x,y).$$

Let $A = A_1 + i A_2$, $A_1, A_2 \in \mathbb{R}$.

Then

$$|u(x,y) - A_1| \leq |f(z) - A|$$



and

$$|v(x,y) - A_2| \leq |f(z) - A|.$$

Work this out.

Using the fact that $f(z) \longrightarrow A$ iff $\operatorname{Re} f(z) \longrightarrow \operatorname{Re} A$, $\operatorname{Im} f(z) \longrightarrow \operatorname{Im}(A)$ we can easily prove:

Theorem: Let f, g be functions defined in a nbhd of z_0, (with z_0 perhaps not in the domain) such that $\lim_{z\to z_0} f(z) = A$ and $\lim_{z\to z_0} g(z) = B$. Then

(1) $\lim_{z\to z_0} (f(z) \pm g(z)) = A \pm B$

(2) $\lim_{z\to z_0} (f(z)\cdot g(z)) = A\cdot B$

(3) If $B \neq 0$, then $\lim_{z\to z_0} \dfrac{f(z)}{g(z)} = \dfrac{A}{B}$

Strategy of proof: Break up everything into its real and imaginary parts, and use results from real variable Calculus

Continuity: Let f be a function defined in a neighbourhood of z_0. We say f is continuous at z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

Let G be an open set in $\mathbb{C}$ and $f: G \to \mathbb{C}$ a function. We say f is continuous on G if f is continuous at each point of G.

Examples:

1. $\lim_{z \to z_0} z = z_0$. (easy. Either use ε-δ, or break up z into its real & imaginary parts.)

2. $\lim_{z \to i} \frac{1}{z} = \frac{1}{i}$, (Just use the theorem above)

3. Let $p(z) = a_0 + a_1 z + \dots + a_n z^n$ be a polynomial. Using the theorem repeatedly, we see that
$$\lim_{z \to z_0} p(z) = p(z_0)$$
and hence p is is continuous on $\mathbb{C}$.

Definition: A rational function is the ratio of two polynomials
$$R(z) = \frac{a_0 + a_1 z + \dots + a_n z^n}{b_0 + b_1 z + \dots + b_m z^m}$$

where $b_0 + b_1 z + \dots + b_m z^m$ is not identically zero. The domain of $R(z)$ is the set of complex numbers on which the denominator

$$b_0 + b_1 z + \dots + b_m z^m$$

does not vanish.

<u>Example</u>: $R(z) = \dfrac{z}{1+z^2}$. ← vanishes at $z = i, -i$.

The domain is $\mathbb{C} \setminus \{i, -i\}$.

Limits and ∞.

We say

$$\lim_{z \to z_0} f(z) = \infty$$

if

$$\lim_{z \to z_0} |f(z)| = \infty.$$

We say

$$\lim_{z \to \infty} f(z) = A$$

if

$$\lim_{w \to 0} f\left(\frac{1}{w}\right) = A.$$

More Examples:

4. $\displaystyle \lim_{z \to i} \frac{z}{z^2+1}$.

Soln: $\lim_{z\to i}\left|\frac{z}{z^2+1}\right| = \lim_{z\to i}\frac{|z|}{|z^2+1|}$

$= \frac{|i|}{|i^2+1|} = \frac{1}{0} = \infty.$

Hence, by defn,

$$\lim_{z\to i}\frac{z}{z^2+1} = \infty$$

5. $\lim_{z\to i}\frac{z-i}{z^2+1} = ?$

Soln: $\frac{z-i}{z^2+1} = \frac{z-i}{(z+i)(z-i)} = \frac{1}{z+i} \longrightarrow \frac{1}{2i}$

as $z\to i$.

6. $\lim_{z\to\infty}\frac{z}{z^2+1} = ?$

Soln: Let $w = \frac{1}{z}$. Then

$$\frac{z}{z^2+1} = \frac{1/w}{\frac{1}{w^2}+1} = \frac{w}{1+w^2} \longrightarrow 0 \text{ as } w\to 0.$$

Ans: $\lim_{z\to\infty}\frac{z}{z^2+1} = 0.$

Exercise: Prove that

$$\lim_{z\to\infty}\frac{a_0+a_1z+\dots+a_nz^n}{b_0+b_1z+\dots+b_mz^m} = \begin{cases}\frac{a_n}{b_m} & \text{if } m=n\\ 0 & \text{if } m>n\\ \infty & \text{if } m<n\end{cases}$$

Proof: Set $w = \frac{1}{z}$.

$$\frac{a_0 + a_1 z + \dots + a_n z^n}{b_0 + b_1 z + \dots + b_m z^m} = \frac{\frac{1}{w^n}\left\{a_0 w^n + a_1 w^{n-1} + \dots + a_{n-1} w + a_n\right\}}{\frac{1}{w^m}\left\{b_0 w^m + b_1 w^{m-1} + \dots + b_{m-1} w + b_m\right\}}$$

$$= w^{m-n}\left\{\frac{a_0 w^n + a_1 w^{n-1} + \dots + a_n}{b_0 w^m + b_1 w^{m-1} + \dots + b_m}\right\}$$

$$= \begin{cases} a_n/b_m & \text{if } m = n \\ 0 & \text{if } m > 0 \\ \infty & \end{cases}.$$

One last example:

7. $f(z) = \bar{z}$ is continuous. Break up f into its real & imaginary parts.

Hence $|z| = \sqrt{x^2+y^2} = \sqrt{z\bar{z}}$ is also continuous. $\leftarrow$ There is a much simpler ε-δ reason (take $\delta = \varepsilon$.)

§2.3 Analyticity

Let $z_0 \in \mathbb{C}$ and let G be a neighbourhood of z_0. A function $f: G \longrightarrow \mathbb{C}$ is differentiable at $z = z_0$ if

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Thought of as a function of Δz.

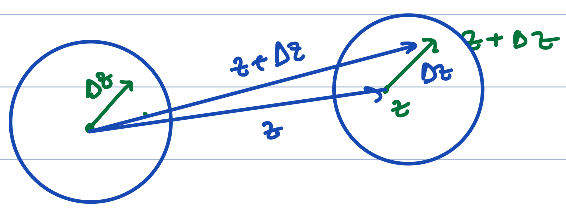
exists.

This limit, if it exists, is called the derivative of f at z_0.

The Difference Quotient:

The ratio $\dfrac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$ is often called the difference quotient of f at z_0

Note that Δz moves in 2-dim'l region.



Examples:

1. $f(z) = \bar{z}$.

The difference quotient is

$$\frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{(\bar{z}+\overline{\Delta z})-\bar{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z}.$$

If Δz is real then the difference quotient is 1, since $\overline{\Delta z} = \Delta z$.

If Δz is purely imaginary, the dif. quotient is -1 (check this).

The limit of $\frac{\overline{\Delta z}}{\Delta z}$ as Δz approaches 0 ~~along~~ in a horizontal direction is 1

. . . . as Δz approaches 0 in a vertical direction, is -1

So the limit does not exist.

<u>Conclusion</u>: $f(z) = \overline{z}$ is nowhere differentiable.

2. $f(z) = z^n$.

Diff. quotient $= \dfrac{(z+\Delta z)^n - z^n}{\Delta z}$

$$= \frac{1}{\Delta(z)} \left\{ z^n + n z^{n-1} \cdot \Delta z + \dots + \binom{n}{i} z^{n-i} (\Delta z)^i + \dots + (\Delta z)^n - z^n \right\}$$

$$= n z^{n-1} + \Delta z \,(\text{stuff})$$

$$\longrightarrow n z^{n-1} \quad \text{as} \quad \Delta z \longrightarrow 0$$

<u>Conclusion</u>: z^n is diff'ble everywhere on $\mathbb{C}$, and its derivative $n z^{n-1}$.

<u>Notations</u>: If f is diff'ble at z_0, we write $f'(z_0)$ or $\frac{df}{dz}(z_0)$ for its derivative at z_0.

What we've shown is $\frac{d}{dz} z^n = n z^{n-1}$.

Theorem:

(1) $\frac{d}{dz}\left(f(z_0) \pm g(z_0)\right) = \frac{df}{dz}(z_0) \pm \frac{dg}{dz}(z_0)$

(2) $\frac{d}{dz}(f \cdot g)(z_0) = f(z_0)\frac{dg}{dz}(z_0) + \left(\frac{df}{dz}(z_0)\right) \cdot g(z_0)$

(3) $\frac{d}{dz}\left(f/g(z_0)\right) = \frac{gf' - fg'}{g^2}$ if $g(z_0) \neq 0$

at z_0.

Definition: Let G be an open set and $f: G \to \mathbb{C}$ a function. We say f is analytic on G (or f is diff'ble on G) if f is differentiable at every point of G.

Let z_0 be a point in $\mathbb{C}$, and f a function defined in a nbhd of z_0. We say f is analytic at z_0 if it is analytic in a neighbourhood of z_0.

Example: Consider $f(z) = |z|^2$. One can check that f is diff'ble at $z = 0$. Since

$$f(z) = z\bar{z}$$

and since $\bar{z}$ is nowhere differentiable, therefore f is not diff'ble at any non-zero number z_0. Indeed, if it were ~~diff~~ dif'ble, then $\bar{z} = \frac{f(z)}{z}$ would be a diff'ble at $z \neq 0$, which is not possible.

So f is diff'ble at 0 but not analytic at 0.

Theorem (Cauchy-Riemann eqns) Suppose f is defined in a neighbourhood of z_0 and is diff'ble at z_0. Write $z_0 = x_0 + iy_0$, $z = x + iy$, $f = u + iv$. Then

(1) $$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)$$

$$= -i\left\{\frac{\partial u}{\partial y}(x_0, y_0) + i\frac{\partial v}{\partial y}(x_0, y_0)\right\}$$

(2) $$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0)$$

$$\frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0)$$

Proof:

Let $\Delta z = \Delta x + i\Delta y$.

Suppose $\Delta z \to 0$ through the real axis.

So $\Delta z = \Delta x$.

Diff quotient: $\dfrac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$

$$= \frac{u(x_0+\Delta x, y_0) - u(x_0, y_0)}{\Delta x}$$

$$+ i\,\frac{v(x_0+\Delta x, y_0) - v(x_0, y_0)}{\Delta x}$$

Since by hypothesis the limit as $\Delta z \to 0$ exists, we see

$$f'(z_0) = \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x}$$

$$+ \lim_{\Delta x \to 0} i \, \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}$$

$$= \frac{\partial u}{\partial x}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0).$$

Next suppose $\Delta z \to 0$ from the imaginary axis. Then $\Delta z = i \Delta y$. Do the same computations to get

$$f'(z_0) = \frac{1}{i} \left\{ \frac{\partial u}{\partial y}(x_0, y_0) + i \frac{\partial v}{\partial y}(x_0, y_0) \right\}.$$

$$= -i \left\{ \frac{\partial u}{\partial y}(x_0, y_0) + i \frac{\partial v}{\partial y}(x_0, y_0) \right\}$$

Rest in next lecture.