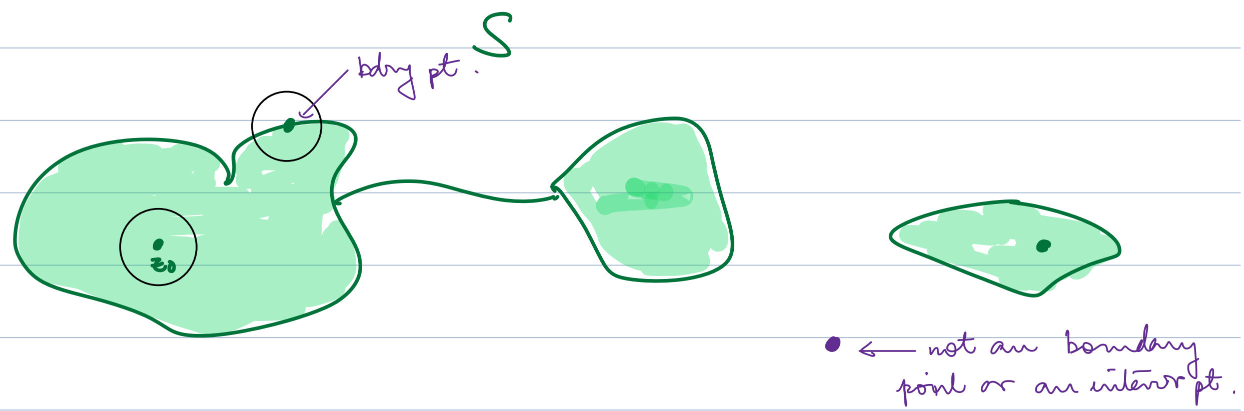
Last time: Defined a circular neighbourhood or an open disk centered at z_0 of radius ε:

$$B_\varepsilon(z_0) = \{ z \in \mathbb{C} \mid |z - z_0| < \varepsilon \}$$

If S is a subset of $\mathbb{C}$ and $z_0 \in S$, then "z_0 is an interior point of S" if $\exists$ $\varepsilon > 0$ s.t. $B_\varepsilon(z_0) \subset S$

"There exists" "such that"



Let $S \subset \mathbb{C}$. A point $z_0 \in \mathbb{C}$ (not necessarily in S!) is said to be a boundary point or a frontier point if every circular neighbourhood of z_0 contains a point in S as well as a point outside S.

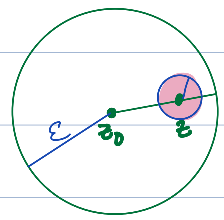
An subset S of $\mathbb{C}$ is called an open set if every point of S is an interior point of S.

A subset of S is called closed if it contains all its boundary points.

Notation: ∂S = boundary of S.

Examples:

1. Open disk $B_\varepsilon(z_0)$. By the triangle inequality $B_\varepsilon(z_0)$ is open. To see this pick $z \in B_\varepsilon(z_0)$.



Let $s = |z - z_0| < \varepsilon$.

Let $0 < r \leq \varepsilon - s$.

Consider $B_r(z) = \{w \in \mathbb{C} \mid |w - z| < r\}$

If $w \in B_r(z)$, then

$$\begin{aligned} |w - z_0| &\leq |w - z| + |z - z_0| \\ &< r + s \\ &\leq (\varepsilon - s) + s \\ &= \varepsilon. \end{aligned}$$

So $|w - z_0| < \varepsilon$, i.e., $w \in B_\varepsilon(z_0)$.

So $B_r(z) \subset B_\varepsilon(z_0)$.

This proves that open disks are open.

(2) Closed disk $S = \{z \in \mathbb{C} \mid |z - z_0| \leq r\}$. This is not open because points on the circumference, i.e. points $z \in \mathbb{C}$ s.t. $|z - z_0| = r$, are in S but are not interior points. However (check this) S is closed. (Check that $\partial S = \{z \in \mathbb{C} \mid |z - z_0| = r\}$)

(4) Punctured disk(s):

$$S = \{z \in \mathbb{C} \mid 0 < |z - z_0| < \varepsilon\}$$

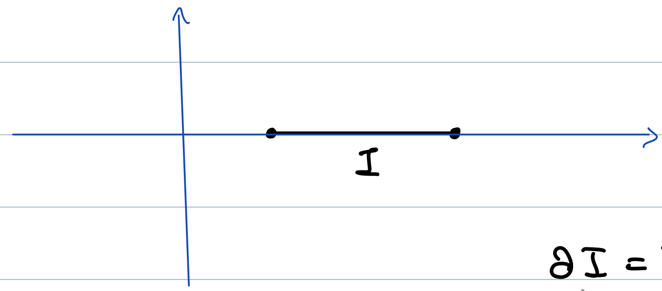
or $$S = \{z \in \mathbb{C} \mid 0 < |z - z_0| \leq \varepsilon\}.$$

In either case (check this!)

$$\partial S = \{z_0\} \cup \{z \in \mathbb{C} \mid |z - z_0| = \varepsilon\}.$$

In the first case, we have an open set (check).

(5) I an interval in $\mathbb{R} \subset \mathbb{C}$.



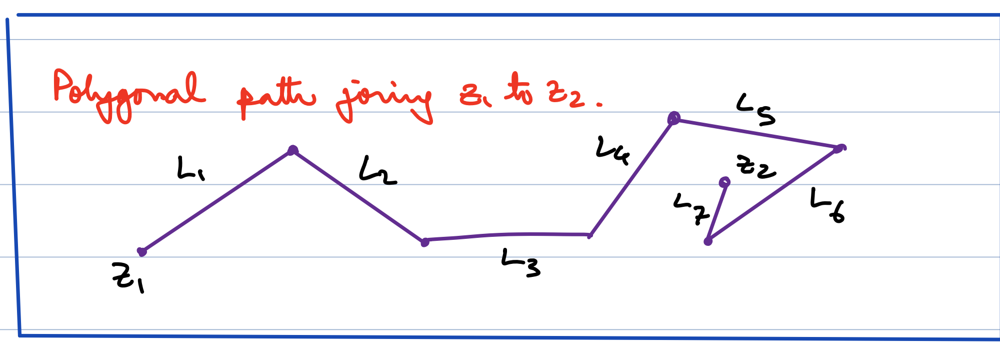
Is it open? (Ans: No!)

Closed (yes if the end points are in I).

$\partial I = ?$

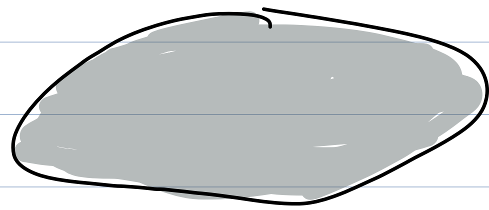
$I \cup \{\text{end-points of } I\}$.

Definition: A polygonal path connecting $z_1 \in \mathbb{C}$ with $z_2 \in \mathbb{C}$ is a finite sequence of line segments, $L_1, L_2, \ldots, L_n$, with L_i and L_{i+1} having a common end-pt.



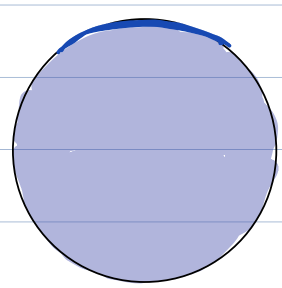
Definition: An open set is said to be connected if any two points in S can be connected by a polygonal path lying entirely in S.

Connected.



Definition: A connected open set is called a domain,

A region is a domain together with some or all or none of its boundary points.



Defn: A set $S \subset \mathbb{C}$ is said to bounded if there exists $R > 0$ such that $|z| < R$ $\forall$ $z \in S$.

"for all"

This is the same as saying

$$S \subset B_R(0)$$

Example: $\mathbb{C}$ is not bounded. $\mathbb{R}$ is not bounded.

The upper half plane $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ is not bdd.

However $B_R(0)$ is bounded.

Functions :

We are interested in complex valued functions

$f: S \longrightarrow \mathbb{C}$ on sets in $\mathbb{C}$ ($S \subseteq \mathbb{C}$).

domain of f.

Range $(f) = \{ z \in \mathbb{C} \mid z = f(s) \text{ for some } s \in S \}$.

Examples:

1. $f(z) = z, \quad z \in \mathbb{C}$.

   Domain of $f = \mathbb{C}$

   Range of $f = \mathbb{C}$

2. $f(z) = \dfrac{z}{z^2+1}$.

   Domain of $f = \mathbb{C} \setminus \{ \pm i \}$.

3. $f(z) = z^2$. Domain of $f = \mathbb{C}$.

Suppose $z = x + iy$, $x, y \in \mathbb{R}$,

and $f(z) = u(z) + i\, v(z)$, $u(z), v(z) \in \mathbb{R}$

$$= u(x,y) + i\, v(x,y)$$

4. $f(z) = z^2$. $u = ?$ $v = ?$

   $z = x + iy \Rightarrow z^2 = x^2 - y^2 - i(2xy)$

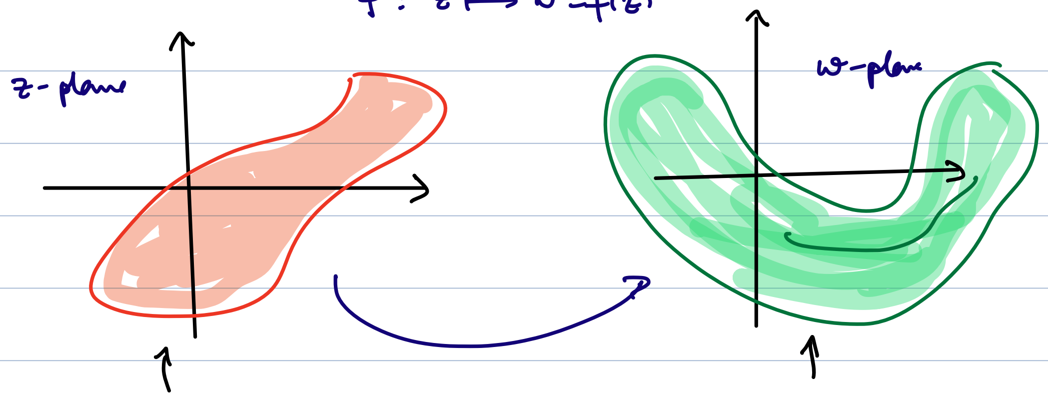
   So $u(x,y) = x^2 - y^2$

   $v(x,y) = 2xy$.

2. $f(x,y) = x + i\,(xy)$

$$u(x,y) = x, \quad v(x,y) = xy.$$

$f : z \longmapsto w = f(z)$

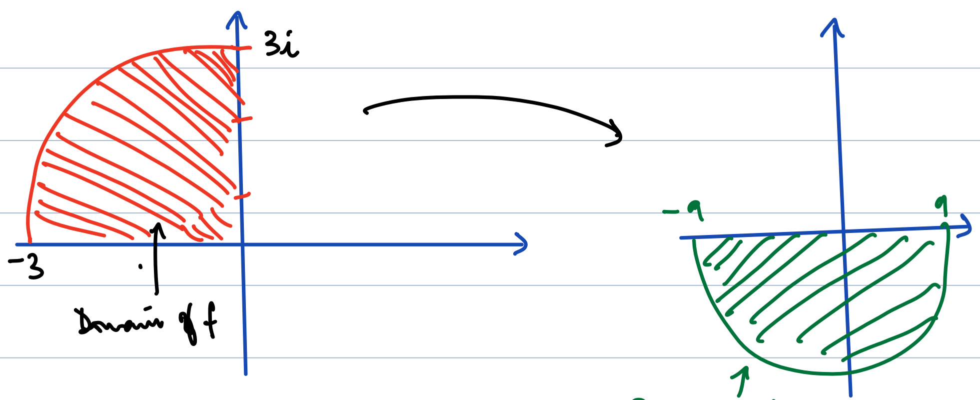


The domain of f

The range of f.

<u>Examples</u> :

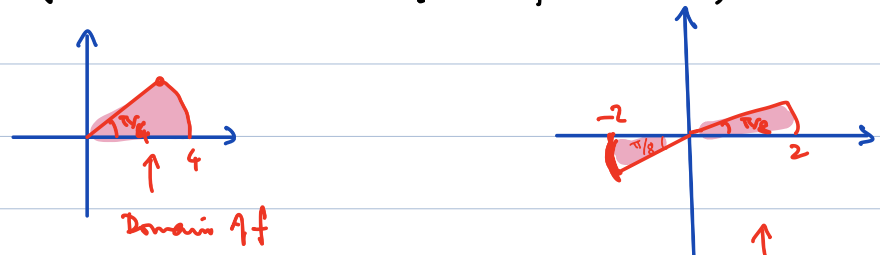
1. $f(z) = z^2$ defined $\left\{ z : |z| \le 3,\ \mathrm{Re}(z) \le 0,\ \mathrm{Im}(z) \ge 0 \right\}$.



Domain of f

Range of f.

2. $f(z) = z^{1/2}$ on $\{ re^{i\theta} \mid 0 \le r \le 4,\ 0 \le \theta \le \pi/4 \}$



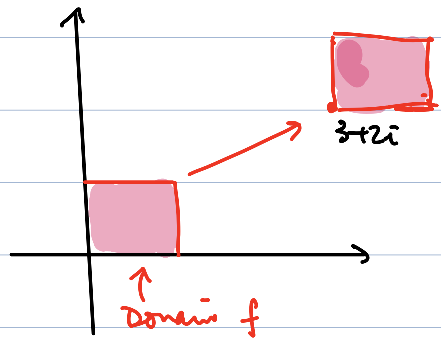
Domain of f

Range of f

range of f.

3. $f(z) = z + 2i + 3$ on the unit square

Unit square $= \{ z \mid 0 \leq Re(z) \leq 1,\ 0 \leq Im(z) \leq 1 \}$



translate everything by the vector $3+2i$

Limits and Continuity :

Definition: Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. We say $\{z_n\}$ converges to $z_0 \in \mathbb{C}$ if $\forall\ \varepsilon > 0$ $\exists\ N \in \mathbb{N}$ s.t.

$$|z - z_0| < \varepsilon$$

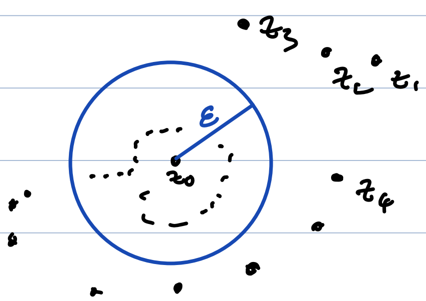
for all $n \geq N$.

In this case we write

$$z_n \longrightarrow z_0 \quad \text{as} \quad n \to \infty$$

Or

$$\lim_{n \to \infty} z_n = z_0.$$



Note: Using the fact that

$$|Re(z)| \leq |z| \text{ and } |Im(z)| \leq |z|$$

we can check that

$$\lim_{n\to\infty} z_n = z_0 \text{ if and only if } \begin{cases} \lim_{n\to\infty} Re(z_n) = Re(z_0) \\ \lim_{n\to\infty} Im(z_n) = Im(z_0) \end{cases}$$

$$|Re(z_n - z_0)| \leq |z_n - z| \;\&\; |Im(z_n - z_0)| \leq |z_n - z_0|.$$

Proof will be supplied at the end of these notes.

Definition: We say $z_n \to \infty$ as $n \to \infty$ if $|z_n| \longrightarrow \infty$ as $n \to \infty$.

Examples:

(a) $z_n = \frac{1}{n} + i \longrightarrow i$ as $n \to \infty$

(b) $\frac{i}{n^2} \longrightarrow 0$ as $n \to \infty$

(c) $z_n = e^{i(2\pi n + \pi/7)} \longrightarrow e^{i\pi/7}$ $\quad (e^{2\pi i n} = 1)$

(d) $z_n = \left(\frac{2-i}{5}\right)^n$.

(d) is interesting:

$$|z_n| = \left|\left(\frac{2-i}{5}\right)^n\right| = \left|\frac{2-i}{5}\right|^n$$

$$= \left(\frac{\sqrt{5}}{5}\right)^n = \left(\frac{1}{\sqrt{5}}\right)^n$$

$$\longrightarrow 0$$

So $|z_n - 0| \to 0$ as $n \to \infty$

i.e. given $\varepsilon > 0$, $\exists$ N s.t

$$|z_n - 0| < \varepsilon \quad \forall\, n \geq N.$$

$$\lim_{n\to\infty} z_n = 0.$$

Definition: Suppose f is a function defined in some neighbourhood of z_0. Then we say

$$\lim_{z \to z_0} f(z) = w_0$$

if for every $\varepsilon > 0$ $\exists\, \delta > 0$ s.t. $\forall z$ with

$$0 < |z - z_0| < \delta$$

we have

$$|f(z) - f(z_0)| < \varepsilon$$

We also say

$$\lim_{z \to z_0} f(z) = \infty$$

if

$$\lim_{z \to z_0} |f(z)| = \infty$$

Extra notes of topics not covered in the Lecture.

(Will cover them in class on Tuesday Jan 25, but do read what is in here.)

Definition: Let z_0 be a point in $S \subset \mathbb{C}$. We say f is continuous if it is continuous at every point of S.

Examples:

1. $\lim_{z \to i} \frac{1}{z}$.

If there is any justice in the world, the answer should be $1/i$.

Why is it $1/i$? Would be great if $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{\lim_{z \to z_0} f(z)}{\lim_{z \to z_0} g(z)}$.

We will come back to this question later.

2. Calculate $\lim_{z \to i} \frac{z-i}{z^2+1}$.

"Solution": $\frac{z-i}{z^2+1} = \frac{z-i}{(z+i)(z-i)} = \frac{1}{z+i} \longrightarrow \frac{1}{i+i} = \frac{1}{2i}$

(provided $z \neq i$ or $-i$)

If there is any justice in the world.

So how much justice is there in the world?
Here is the required theorem.

Theorem: Suppose

$$\lim_{z \to z_0} f(z) = A$$

and

$$\lim_{z \to z_0} g(z) = B.$$

Then

(a) $$\lim_{z \to z_0} \left(f(z) \pm g(z) \right) = A \pm B$$

(b) $$\lim_{z \to z_0} \left(f(z) \cdot g(z) \right) = A \cdot B$$

(c) If $B \neq 0$, $$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{A}{B}.$$

Corollary: If f and g are continuous at z_0 then $f \pm g$, fg, f/g are continuous at z_0 (for the statement about f/g, we require that $g(z_0) \neq 0$).

Examples: Recall that a polynomial is a function p

given by a formula of the following kind

$$p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n,$$

where $a_0, a_1, \dots, a_n$ are constant complex numbers.

A rational function is a function of the following form

$$f(z) = \frac{p(z)}{q(z)} = \frac{a_0 + a_1 z + \dots + a_n z^n}{b_0 + b_1 z + \dots + b_m z^m}, \quad a_i, b_j \in \mathbb{C}.$$

where p and q are polynomial functions and q is not identically zero.

So here are examples of continuous functions.

(1) $f(z) \equiv c$, c a constant, $z \in \mathbb{C}$. Then f is continuous on $\mathbb{C}$. (Note that $\lim_{z \to z_0} f(z) = \lim_{z \to z_0} c = c = f(z_0)$.)

(2) $f(z) = z$, $z \in \mathbb{C}$. Then f is continuous on $\mathbb{C}$.

(Given $\varepsilon > 0$, pick $\delta = \varepsilon$. Then $|z - z_0| < \varepsilon$ whenever $0 < |z - z_0| < \delta$, i.e., $\lim_{z \to z_0} z = z_0$, which in turn means $\lim_{z \to z_0} f(z) = f(z_0)$.)

(3) From the theorem it follows that polynomials are continuous on $\mathbb{C}$, since a polynomial is a combination of sums and products of examples (1) and (2): $p(z) = a_0 + a_1 z + \dots + a_n z^n$.

(4) Once again, from the theorem and the definition of a rational function a rational function

$$f(z) = \frac{p(z)}{q(z)} = \frac{a_0 + a_1 z + \dots + a_n z^n}{b_0 + b_1 z + \dots + b_m z^m}$$

is continuous on the set $\mathbb{C} \setminus \{\text{zeroes of } q(z)\}$.

More examples:

1. $\displaystyle\lim_{z\to i} \frac{1}{z} = \frac{1}{i}$

2. $\displaystyle\lim_{z\to i} \frac{z}{z^2+1}$. This is not defined at $z=i$.

   Now $\displaystyle\left|\frac{z}{z^2+1}\right| = \frac{|z|}{|z^2+1|} \longrightarrow \frac{1}{\lim_{z\to i}|z^2+1|} = \infty$ as $z\to i$.

3. $\displaystyle\lim_{z\to i} \frac{z-i}{z^2+1}$.

   This is $\underline{\text{not}}$ defined at $z=\pm i$!

   Define a new function

$$\tilde{f} : \mathbb{C}\setminus\{i\} \longrightarrow \mathbb{C}$$

   by the rule

$$\tilde{f}(z) = \begin{cases} \dfrac{1}{z+i}, & z \neq i, -i \\[2ex] \dfrac{1}{2i}, & z \neq -i. \end{cases}$$

Then $\tilde{f}$ is continuous on $\mathbb{C}\setminus\{i\}$. In other words "f has a 'removable singularity' at i"! We will define 'removable singularity' more formally later in the course.

Definition: $\displaystyle\lim_{z\to\infty} f(z) = \lim_{w\to 0} f\left(\frac{1}{w}\right)$.

4. (a) $\displaystyle\lim_{z\to\infty} \frac{z}{z^2+1} = \lim_{w\to 0} \frac{1/w}{1/w^2+1}$

$$= \lim_{w\to 0} \frac{w}{1+w^2}$$

$$= 0.$$

(b) $\lim\limits_{z \to \infty} \dfrac{iz-2}{4z+i} = i/4$

Check: Set $w = 1/z$

$$\frac{iz+2}{4z+i} = \frac{i/w+2}{4/z+i} = \frac{i+2w}{4+iz} \longrightarrow \frac{i}{4} \text{ as } w \to 0.$$

Exercise: Show that

$$\lim_{z \to \infty} \frac{a_0 + a_1 z + \ldots + a_n z^n}{b_0 + b_1 z + \ldots + b_m z^m} = \begin{cases} \frac{a_n}{b_m} & \text{if } n = m \\ 0 & \text{if } n < m \\ \infty & \text{if } n > m \end{cases}$$

Solution: Set $w = 1/z$.

$$a_0 + a_1 z + \ldots + a_n z^n = \frac{1}{w^n}\left(a_0 w^n + a_1 w^{n-1} + \ldots + a_{n-1} w + a_n\right)$$

and

$$b_0 + b_1 z + \ldots + b_m z^m = \frac{1}{w^m}\left(b_0 w^m + b_1 w^{m-1} + \ldots + b_{m-1} w + b_m\right)$$

It follows that

$$\frac{a_0 + a_1 z + \ldots + a_n z^n}{b_0 + b_1 z + \ldots + b_m z^m} = w^{m-n} \frac{a_0 w^n + \ldots + a_{n-1} w + a_n}{b_0 w^m + \ldots + b_{m-1} w + b_m}$$

$$\xrightarrow{\text{as } w \to 0} \lim_{w \to 0} \left(w^{m-n}\right) \cdot \frac{a_n}{b_m}$$

$$= \begin{cases} \frac{a_n}{b_m} & \text{if } n = m \\ 0 & \text{if } n < m \\ \infty & \text{if } n > m. \end{cases}$$