Jan 18, 2022 Lecture 3 MATH 300

Last Time: Polar coordinates.

If $z \neq 0$, then can write

(*) $z = r e^{i\theta}$ $\quad r>0, \ \theta \in \mathbb{R}$

This is the polar form of z. Here

$$r = |z| = \{Re(z)^2 + Im(z)^2\}^{1/2}.$$

The θ's satisfying (*) form a set denoted $\arg(z)$.

If $\theta_1, \theta_2 \in \arg(z)$ then

$$\theta_2 - \theta_1 = 2\pi k, \quad k \in \mathbb{Z}.$$

Conversely, if $\theta_1 \in \arg(z)$ and $\theta_1 - \theta_2 = 2\pi k$ for some integer k, then θ_2 also lies in $\arg(z)$. Indeed

$$e^{i(\theta_2 - \theta_1)} = \frac{e^{i\theta_2}}{e^{i\theta_1}} = \frac{e^{i(\theta_1 + 2\pi k)}}{e^{i\theta_1}}$$

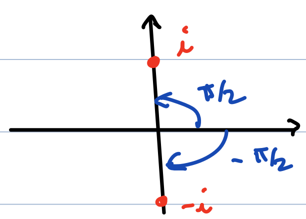
$$= \frac{\cancel{e^{i\theta_1}}}{\cancel{e^{i\theta_1}}} \cdot e^{i2\pi k}$$

$$= 1$$

So $e^{i\theta_2} = e^{i\theta_1}$ whence

$$r e^{i\theta_2} = r e^{i\theta_1} = z.$$

$\operatorname{Arg}(z)$ is the unique complex number in $\arg(z) \cap (-\pi, \pi]$.



$\operatorname{Arg}(i) = \pi/2$

$\operatorname{Arg}(-i) = -\pi/2$.

$\cos\theta + i \sin\theta$

$\cos\theta_2 + i\sin\theta_2$

Multiplication as rotation :

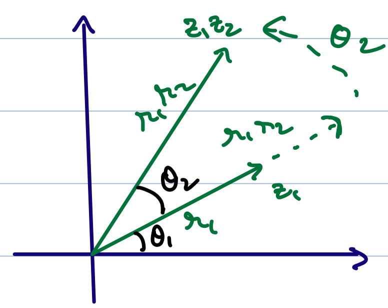
Let $z_1, z_2 \in \mathbb{C} - \{0\}$, $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$.

Hence

$$z_1 \cdot z_2 = r_1 r_2 \, e^{i\theta_1} \cdot e^{i\theta_2} = r_1 r_2 \, e^{i(\theta_1+\theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \, e^{i(\theta_1-\theta_2)}.$$

- $z_1 z_2$ is obtained by taking the vector z_1, scaling its length by a factor of r_2, and then rotating the vector by an angle of θ_2.



Note : 1. Multiplying by i amounts to rotating by $\pi/2$ without altering the length.

2. Multiplying by $e^{i\theta}$ amounts to rotating by an angle of θ without altering the length
$\left(e^{i\theta} = \cos\theta + i\sin\theta \Rightarrow |e^{i\theta}| = \sqrt{\cos^2 + \sin^2} = 1. \right)$

- z_1/z_2 amounts to scaling z_1 by a factor of $\frac{1}{r_2}$ and rotating the resulting vector by $-\theta_2$.

D' Moivre's formula:

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)} \quad n\in\mathbb{Z}$$

"Pf":

$$(e^{i\theta})^n = e^{i(n\theta)} \quad //$$

Roots of unity:

Problem: Let m be a positive integer. Find all the m^{th} roots of 1, i.e., find all the solutions of

$$z^m = 1.$$

Example: Find all solutions of $z^4 = 1$.

Soln:

Know $$e^{(2\pi i)n} = 1, \quad n\in\mathbb{Z}.$$

So $$(e^{2\pi i})^{n/4} = \cos\left(\frac{2\pi n}{4}\right) + i\sin\left(\frac{2\pi n}{4}\right), \quad n\in\mathbb{Z}$$

is a fourth root of 1. In greater detail,

$$\left\{e^{2\pi i\left(\frac{n}{4}\right)}\right\}^4 = e^{2\pi i n} = 1 \quad \forall\, n\in\mathbb{Z}.$$

Let

$$\omega_4 = e^{2\pi i/4}.$$

Then $$\omega_4^4 = 1.$$

$$\begin{array}{ccccccc}
1 = \omega_4^0 & , & \omega_4 & , & \omega_4^2 & , & \omega_4^3 \\
\| & & \| & & \| & & \| \\
1 = \omega_4^4 & , & \omega_4^5 & , & \omega_4^6 & , & \omega_4^7 \\
 & & \| & & & & \\
1 = \omega_4^8 & , & \omega_4^9 & , & \omega_4^{10} & , & \omega_4^{11}
\end{array}$$

More formally, if $\eta \in \mathbb{C}$ such that

$$\eta^4 = 1,$$

then $|\eta| = 1$, so $\eta = e^{i\theta}$ for some θ.

Hence

$$(e^{i\theta})^4 = 1,$$

i.e. $$e^{i(4\theta)} = 1$$

So $$4\theta \in \arg(1) = \{2\pi n \mid n \in \mathbb{Z}\}.$$

Hence $4\theta = 2\pi n$ for some n,

i.e. $$\theta = 2\pi \left(\frac{n}{4}\right)$$

We can write n as

$$n = \underbrace{4k}_{\text{integer}} + r, \quad r \in \{0, 1, 2, 3\}$$

remainder upon dividing n by 4.

$$\eta = e^{i\theta} = e^{i(2\pi(\frac{n}{4}))}$$

$$= e^{i\{2\pi(4k+r)\}/4} \qquad r \in \{0, 1, 2, 3\}$$

$$= e^{i(2\pi k + r/4)}$$

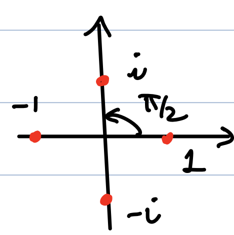
$$= e^{i2\pi k} \cdot e^{ir/4}$$

$$= e^{ir/4} = \omega_4^r, \text{ and } r = \{0, 1, 2, 3\}$$

The only solutions are

$$1, \omega_4, \omega_4^2, \omega_4^3.$$

$$\omega_4 = e^{2\pi i/4} = e^{i(\pi/2)} = i.$$

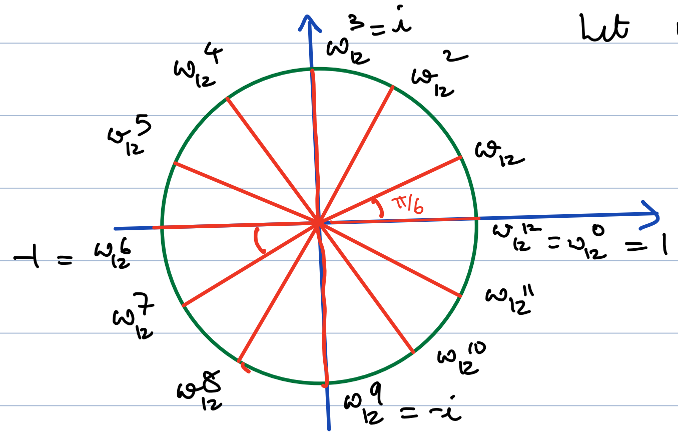


$$\omega_4^2 = i^2 = -1$$

$$\omega_4^3 = i^3 = -i$$

$$\omega_4^4 = \omega_4^0 = 1$$

2. The same analysis for $z^{12} = 1$ gives:



Let $\omega_{12} = e^{i(2\pi)/12}$
$= e^{i\pi/6}$

Here is the general solution of

$$z^m = 1 \quad \text{———————} \quad (*)$$

where m is a positive integer (we've done $m = 4, 12$)

Set

$$\omega_m = e^{2\pi i/m}.$$

$$= \cos\left(\frac{2\pi}{m}\right) + i \sin\left(\frac{2\pi}{m}\right).$$

Then the distinct solutions of $(*)$ are

$$\underbrace{1 = \omega_m^0, \ \omega_m, \ \omega_m^2, \ \ldots, \ \omega_m^{m-1}}_{m \text{ solutions.}}$$

Check that $z^m - 1 = (z-1)(z-\omega_m)(z-\omega_m^2)\cdots(z-\omega_m^{m-1})$

<u>Example</u>: Show that $1 + \omega_m + \omega_m^2 + \ldots + \omega_m^{m-1} = 0$

<u>Solution</u>: Recall that for number x (geometric series)

$$1 + x + x^2 + \ldots + x^{m-1} = \frac{1 - x^m}{1 - x}.$$

Therefore

$$1+\omega_m+\omega_m^2+\dots+\omega_m^{m-1} = \frac{1-\omega_m^m}{1-\omega_m}$$

$$= \frac{1-1}{1-\omega_m}$$

$$= 0 \quad //$$

Example: Let $z \in \mathbb{C} \setminus \{0\}$ and m a positive integer. Solve the equation

$$\zeta^m = z \leftarrow \text{known} \qquad (\zeta \text{ is the unknown})$$

for ζ.

Solution: Let

$$z = re^{i\theta}$$

be a polar form of z.

Let $\eta = r^{1/m} e^{i\theta/m}$.

Check that

$$\zeta = \eta = \omega_m^0 \cdot \eta,\ \zeta = \omega_m \eta,\ \dots,\ \zeta = \omega_m^{m-1}\eta$$

are the only solutions of $\zeta^m = z$.

Further explanation: Let $\zeta = \eta'$ be another soln of $\zeta^m = z$. Then

$$(\eta')^m = re^{i\theta}.$$

It follows that $|\eta'| = r^{1/m}$, so

$$\eta' = r^{1/m} e^{i\phi}$$

for some ϕ.

$$z = re^{i\theta} = (\eta')^m = \left(r^{1/m} e^{i\phi}\right)^m = re^{i(m\phi)}$$

Thus $m\phi \in \arg(z)$. So

$$m\phi = \theta + 2\pi n \qquad \text{for some } n \in \mathbb{Z}.$$

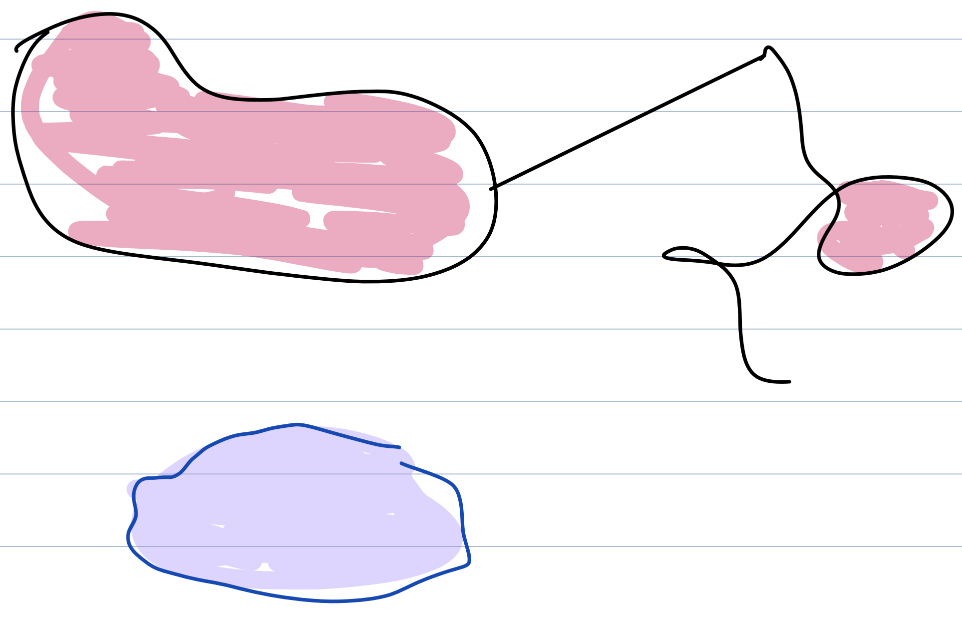
Hence $\phi = \dfrac{\theta}{m} + \dfrac{2\pi n}{m}$.

$$\eta' = r^{1/m} e^{i\phi} = r^{1/m} e^{i\left(\theta/m + \frac{2\pi n}{m}\right)}$$
$$= r^{1/m} e^{i\theta/m} \cdot e^{i(2\pi n/m)}$$
$$= \eta \cdot \omega_m^n$$

But $\omega_m^n = \omega_m^k$, for some $k \in \{0, 1, \ldots, m-1\}$.

So $\eta' = \eta \cdot \omega_m^k$, for some $k \in \{0, 1, \ldots, m-1\}$

Planar Sets

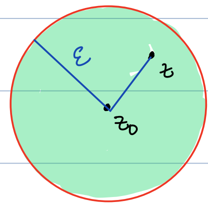


Definitions:

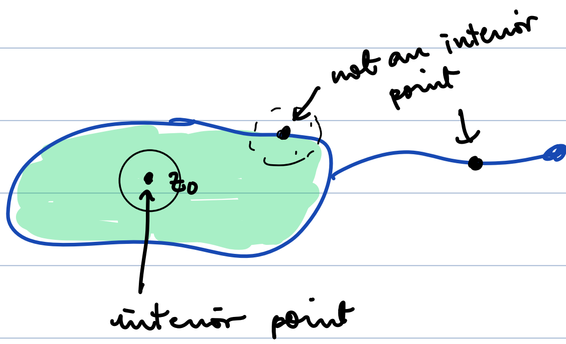
(1) Let $z_0 \in \mathbb{C}$, and ε a positive real number ($\varepsilon > 0$)

$$B_\varepsilon(z_0) = \{ z \in \mathbb{C} \mid |z - z_0| < \varepsilon \}$$

This is called the open disk of radius ε centred at z_0.



(2) Let $S \subset \mathbb{C}$ and $z_0 \in S$. We say z_0 is an interior point of S if $\exists\, \varepsilon > 0$ such that $B_\varepsilon(z_0) \subset S$.



(3) A boundary point or frontier point of S (S as above) is a point such that every open ball centred at that point contains at least one point of S and at least one point outside S

(4) If every point of S is an interior point, we call S an open set.

Examples: 1. A closed disk, i.e. $\{ z \in \mathbb{C} \mid |z - z_0| \leq \varepsilon \}$, is not open. It contains $B_\varepsilon(z_0)$ and the bounding circle. The bounding circle is the boundary, i.e. the set of boundary points.

2. The open disk $B_\varepsilon(z_0)$ is open. This requires the triangle inequality (try proving it yourself using the Δ-ineq.)

3. The punctured disk $\{z\in\mathbb{C} \mid 0<|z-z_0|<\varepsilon\}$ has as boundary $C\cup\{z_0\}$, where C is the bounding circle $\{z\in\mathbb{C} \mid |z-z_0|=\varepsilon\}$

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