LECTURE 22

Date of Lecture: April 7, 2022

For $r > 0$ and $a \in \mathbb{C}$, $B_r(a)$ will denote the open disc of radius r centred at a, and $C_r(a)$ the bounding circle of $B_r(a)$. In case $r = 0$, we simplify the notation and write B_r and C_r for $B_r(a)$ and $C_r(a)$.

1. **Residues**

See also § 6.1 of the textbook.

1.1. **Isolated singularities and residues.** Let f be an analytic function on a domain D with an isolated singularity at z_0. Recall from Lecture 20 that this means there is an open disc $B_\rho(z_0)$ centred at z_0 such that the domain D contains the punctured disc $B_\rho(z_0) \smallsetminus \{0\}$. Recall from that lecture that there are three kinds of isolated singularities, namely, (a) removable singularities; (b) poles; and (c) essential singularities. It is clear from the definitions that f has a pole of order n at z_0 if and only if $(z-z_0)^n f(z) = g(z)$ has a removable singularity at z_0, in other words $(z-z_0)f(z)$ can be extended to an analytic function $g(z)$ on the domain $D \cup \{z_0\}$. In fact if

$$f(z) = \sum_{j=-n}^{\infty} a_j (z-z_0)^j \tag{1.1.1}$$

is the Laurent expansion of f in $B_\rho(z_0)$, then $g(z_0) = a_{-n}$.

In the above situation, the *residue of f at z_0* is defined to be a_{-1}. We denote this residue by the symbol $\mathrm{Res}(f; z_0)$ or simply as $\mathrm{Res}(z_0)$. Thus

$$\mathrm{Res}(f; z_0) = a_{-1}. \tag{1.1.2}$$

It is clear from the Laurent expansion (1.1.1) and the fact that $a_j(z-z_o)^j$ has an anti-derivative in $\mathbb{C} \smallsetminus \{z_0\}$ for all j except $j = -1$ that

$$\mathrm{Res}(f; z_0) = \frac{1}{2\pi i} \oint_{C_r(z_0)} f(z)dz \tag{1.1.3}$$

where r is a positive real number less that ρ, i.e. $0 < r < \rho$, so that $C_r(z_0)$ lies in the punctured disc $B_\rho(z_0) \smallsetminus \{z_0\}$.

Suppose f has a pole or a removable singularity at z_0. Then there exists a positive integer n such that $(z-z_0)^n f(z)$ has a removable singularity, i.e. there is an analytic fucntion g on $D \cup \{z_0\}$ such that $(z-z_0)^n f(z) = g(z)$ on D. In this case we have $\oint_{C_r(z_0)} f(z)dz = \oint_{C_r(z_0)} g(z)/(z-z_0)^n dz = (2\pi i)((n-1)!)^{-1} g^{(n-1)}(z_0)$.

To summarise the last assertion, *if n is a positive integer such that $(z-z_0)f(z)$ has a removable singularity at z_0* then

$$\mathrm{Res}(f; z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \{(z-z_0)^n f(z)\}. \tag{1.1.4}$$

The 0^{th} derivative of a function is interpreted as the function itself. Thus when $n = 1$, the right side of the above formula is the limit of $(z-z_0)f(z)$ as $n \to z_0$.

The following theorem is a generalisation of the formula in (1.1.3). It is called the *Cauchy Residue Theorem*.

Theorem 1.1.5. (The Cauchy Residue Theorem *Let Γ be a simple loop oriented positively, and $z_1, \dots, z_k$ points in the interior of Γ. Suppose f is analytic on Γ and in the interior of Γ, except at $z_1, \dots, z_k$. Then*

$$\int_\Gamma f(z)dz = 2\pi i \sum_{j=1}^{k} \operatorname{Res}(f; z_j).$$

Proof. Let C_j, $j = 1, \dots, k$ be a circle in the interior of Γ centred at z_j, with $C_i \cap C_j = \emptyset$ for $i \neq j$. Let Γ_1 and Γ_2 be loops as shown in the FIGURE 1. Then for any continuous function ϕ on $\Gamma_1 \cup \Gamma_2$, it is clear that $\int_{\Gamma_1} \phi(z)dz + \int_{\Gamma_2} \phi(z)dz = \int_\Gamma \phi(z)dz - \sum_{j=1}^{k} \oint_{C_j} \phi(z)dz$. Thus

$$\int_\Gamma f(z)dz - \sum_{j=1}^{k} \oint_{C_j} f(z)dz = \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz = 0.$$

The last equality is due to the fact that f is analytic on the loops Γ_1 and Γ_2, as well as in their interiors.

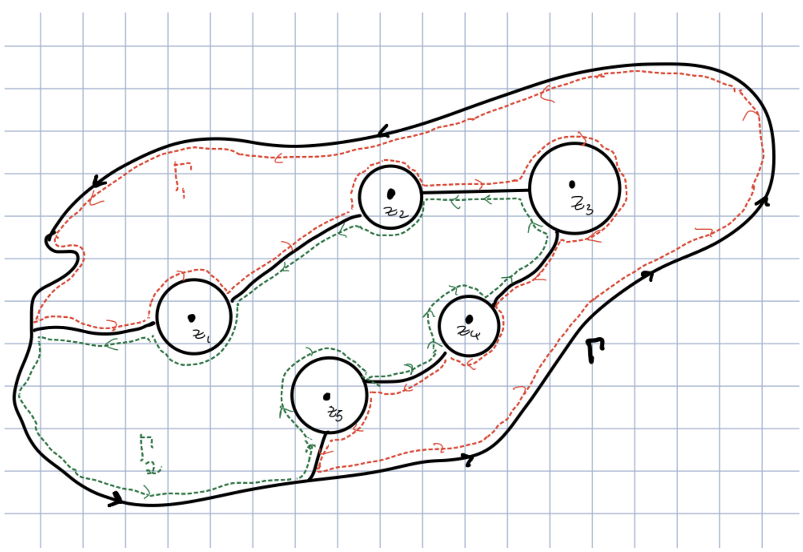

FIGURE 1. Γ is the outer loop in black, oriented positively. The dotted loop in red indicates the loop Γ_1 and its direction, and the one in green does the same for Γ_2. The circles are the various C_j, for $j = 1, \dots, k$. In the picture $k = 5$.

Thus

$$\int_\Gamma f(z)dz = \sum_{j=1}^{k} \oint_{C_j} f(z)dz.$$

The required result now follows from (1.1.3). $\square$

1.2. **Examples.** Here are some examples. See §6.1 of the textbook for more examples of residues and applications of the Cauchy Residue Theorem. Also try out exercises at the end of the section.

1. $f(z) = e^{1/z}$ has an isolated singularity at $z = 0$. The singularity is an essential singularity. The Laurent expansion around $z = 0$ is $\sum_{n=0}^{\infty} z^{-n}/n!$.

2. The functions $\dfrac{e^z - 1}{z}$, $\dfrac{\sin z}{z}$, and, $\dfrac{1 - \cos z}{z}$ have removable singularities at $z = 0$. The Maclaurin series for these functions are $\sum_{n=0}^{\infty} \frac{1}{(n+1)!} z^n$, $\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} z^{2n}$, and $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{(2n)!} z^{2n-1}$, respectively.

3. Find all the residues of $f(z) = \dfrac{1}{z^2(z-2)(z-5)}$. Evaluate $\oint_{|z|=3} f(z)dz$.

Solution: We will use the formula is (1.1.4). The function has isolated singularities is $z = 0$, $z = 2$, and $z = 5$, According to that formula

$$\begin{aligned}
\operatorname{Res}(f;0) &= \lim_{z\to 0} \frac{d}{dz}\left\{\frac{1}{(z-2)(z-5)}\right\} = \lim_{z\to 0} \frac{-(z-2)-(z-5)}{(z-2)^2(z-5)^2} = \frac{7}{100} \\
\operatorname{Res}(f;2) &= \lim_{z_0\to 2} \frac{1}{z^2(z-5)} = -\frac{1}{12} \\
\operatorname{Res}(f;5) &= \lim_{z_0\to 5} \frac{1}{z^2(z-3)} = \frac{1}{50}
\end{aligned}$$

To evaluate $\oint_{|z|=3} f(z)dz$, note that the only singularities of f in the interior of C_3 are $z = 0$ and $z = 2$. It follows from Cauchy's Residue Theorem that

$$\oint_{|z|=3} f(z)dz = 2\pi i((\operatorname{Res}(f;0) + \operatorname{Res}(f;2)) = 2\pi i\left(\frac{7}{100} - \frac{1}{12}\right) = -\frac{2\pi i}{75}$$

4. Let $f(z) = \dfrac{1}{z\sin z}$ and $g(z) = \dfrac{1}{z^2 \sin z}$. Find $\operatorname{Res}(f;0)$ and $\operatorname{Res}(g;0)$.

Solution: Since $\sin z$ has a simple zero at $z = 0$, $z\sin z$ has a zero of order two at $z = 0$, and $z^2 \sin z$ has a zero of order three at $z = 0$. These are isolated zeros. Therefore $f(z)$ has a pole of order 2 and $z = 0$ and $g(z)$ has a pole of order 3 at $z = 0$. The functions f and g have no other signularity in $\mathbb{C}$. In Example 2 above we showed that $\sin z/z$ has a removable singularity at $z = 0$ and its Maclaurin series is $\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} z^{2n}$. Define

$$h(z) = \begin{cases} \dfrac{z}{\sin z} & \text{if } z \neq 0 \\ 1 & \text{if } z = 0. \end{cases}$$

Then $h(z)$ is analytic on $\mathbb{C}$. We then have

$$f(z) = \frac{h(z)}{z^2} \qquad \text{and} \qquad g(z) = \frac{h(z)}{z^3}.$$

From (1.1.4) we see that

$$\operatorname{Res}(f;0) = h'(0) \qquad \text{and} \qquad \operatorname{Res}_(g;0) = \frac{h''(0)}{2}.$$

It is somewhat complicated to follow the above recipe for computing residues. Here is an alternate way of doing this. Let $w = z^2$. Then

$$\frac{\sin z}{z} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} w^n.$$

Let $\sum_{n=0}^{\infty} b_n w^n$ be a power series such that

$$\left(\sum_{n=0}^{\infty} b_n w^n \right) \left(\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} w^n \right) = 1,$$

Then, clearly $h(z) = \sum_{n=0}^{\infty} b_n z^{2n}$. Now $\mathrm{Res}(f;0) = \mathrm{Res}(h(z)/z^2;0)$ is the coefficient of z in the Maclaurin's expansion of $h(z)$, and since the Maclaurin's series for $h(z)$ has only even powers,

$$\mathrm{Res}(f;0) = 0.$$

To compute $\mathrm{Res}(g;0) = \mathrm{Res}(h(z)/z^3;0)$, we have to look for the coefficient of z^2 in the Maclaurin's expansion of $h(z)$, and this is b_1. We can compute b_1 as follows. We have

$$(b_0 + b_1 w + b_2 w^2 + \dots)(1 - \tfrac{1}{6} w + \tfrac{1}{120} w^2 - \dots) = 1$$

It follows that $b_0 = 1$ and $b_0(-1/6) + b_1 = 0$, giving $b_1 = 1/6$. Thus

$$\mathrm{Res}(g;0) = \frac{1}{6}.$$

5. Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{13 - 5\cos\theta} d\theta$. (See also Example 2 of Lecture 21.)

Solution: We have $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$. Let $z = e^{i\theta}$, so that $d\theta = \dfrac{dz}{iz}$. Then

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{13 - 5\cos\theta} d\theta &= \oint_{C_1} \frac{dz}{(iz)\{13 - \frac{5}{2}(z + (1/z)\}} \\
&= \frac{2}{i} \oint_{C_1} \frac{dz}{26z - 5z^2 - 5} \\
&= 2i \oint_{C_1} \frac{dz}{5z^2 - 26z + 5} \\
&= \frac{2i}{5} \oint_{C_1} \frac{dz}{(z-5)(z - \frac{1}{5})}
\end{aligned}$$

Of the two poles of $\dfrac{1}{(z-5)(z-\frac{1}{5})}$, only the one at $z=\frac{1}{5}$ lies in the interior of C_1. By the Cauchy Residue Formula, we therefore have

$$\begin{aligned}\int_0^{2\pi} \frac{d\theta}{13-5\cos\theta} d\theta &= \frac{2i}{5}(2\pi i)\mathrm{Res}\left(\frac{1}{(z-5)(z-\frac{1}{5})}, 0\right) \\ &= -\frac{4}{5}\pi \lim_{z\to 1/5} \frac{1}{z-5} \\ &= -\frac{4\pi}{5(\frac{1}{5}-5})\\ &= \frac{\pi}{6}.\end{aligned}$$

The next example does not involve residues.

6. Suppose f is an entire function and there are positive real numbers R_0 and M such that $|f(z)| \le M|z|^n$ for all z with $|z| > R_0$. Show that f is a polynomial of degree at most n.

Solution: By the Cauchy estimates (see (2.1.1) of Lecture 16, or Theorem 20 on page 215 of the textbook), we have for all $k > 0$,

$$|f^{(n+k)}(0)| \le \frac{n!MR^n}{R^{n+k}} = \frac{n!M}{R^k}.$$

Since $k > 0$, the quantity on the right approaches 0 as $R \to \infty$. Thus $|f^{(n+k)}(0)| = 0$ for $k > 0$, $f^{(m)}(0) = 0$ for all $m > n$. It follows that the Maclaurin's series for f is

$$f(z) = \sum_{m=0}^{n} f^{(m)}(0)z^m,$$

which means f is a polynomial of degree less than or equal to n (the degree equals n if $f^{(n)}(0) \neq 0$).