LECTURE 21

Date of Lecture: April 5, 2022

For $r > 0$ and $a \in \mathbb{C}$, $B_r(a)$ will denote the open disc of radius r centred at a, and $C_r(a)$ the bounding circle of $B_r(a)$. In case $r = 0$, we simplify the notation and write B_r and C_r for $B_r(a)$ and $C_r(a)$.

1. **Examples**

1. Let $f(z) = \dfrac{1}{(3z-1)(z+2)}$.

- **(a)** Find the Laurent expansion of $f(z)$ centred at 0 for $|z|$ large. What is the region of convergence?
- **(b)** Find the Laurent expansion of $f(z)$, centred at 0 , in an annular region containing $|z| = 1$. What is the region of convergence?
- **(c)** Find the Maclaurin series for $f(z)$. What is the region of convergence?

Solution: We will repeatedly use the fact that

$$(\#) \qquad \frac{1}{1-w} = \sum_{n=0}^{\infty} w^n \qquad \text{if } |w| < 1.$$

The partial fraction decomposition of $f(z)$ is

$$f(z) = \frac{1}{7}\left(\frac{3}{3z-1} - \frac{1}{z+2}\right).$$

Let us compute the Maclaurin and the Laurent series for each of the two partial fractions above. Let us first do the computations for $3/(3z-1)$. If $|z| > 1/3$, then $|1/3z| < 1$ and hence, by setting $w = 1/3z$ in (#) we get $\frac{3}{3z-1} = \frac{3}{3z(1-(1/3z))} = \frac{1}{z} \cdot \frac{1}{1-(3/z)} = \frac{1}{z}\sum_{n=0}^{\infty} \frac{3^n}{z^n} = \sum_{n=0}^{\infty} \frac{3^n}{z^{n+1}}$.

Thus

$$(*) \qquad \frac{3}{3z-1} = \sum_{n=1}^{\infty} \frac{3^{n-1}}{z^n}, \qquad \text{if } |z| > 1/3.$$

On the other hand, if $|z| < 1/3$, then $|3z| < 1$, and therefore, setting $w = 3z$ in (#) we get $\frac{3}{3z-1} = -\frac{3}{1-3z} = -3\sum_{n=0}^{\infty}(3z)^n$. Thus

$$(**) \qquad \frac{3}{3z-1} = -\sum_{n=0}^{\infty} 3^{n+1} z^n, \qquad \text{if } |z| < 1/3.$$

Next let us do our computations for the $1/(z+2)$. If $|z| > 2$, then $|2/z| < 1$, and hence, by setting $w = 2/z$ in (#) we get $\frac{1}{z+2} = \frac{1}{z} \cdot \frac{1}{1-(-z/2)} = \frac{1}{z}\sum_{n=0}^{\infty} \frac{(-2)^n}{z^n} = \sum_{n=0}^{\infty} \frac{(-2)^n}{z^{n+1}}$. In other words

$$(\dagger) \qquad \frac{1}{z+2} = \sum_{n=1}^{\infty} \frac{(-2)^{n-1}}{z^n}, \qquad \text{if } |z| > 2.$$

If $|z|<2$, then $|z/2|<1$, and so applying (#) to $w=z/2$ we get $\frac{1}{z+2}=\frac{1}{2}\cdot\frac{1}{1-(-z/2)}=\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{2^n}$. It follows that

$$(\ddagger)\qquad \frac{1}{z+2}=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}z^n, \qquad \text{if } |z|<2.$$

(a) If $|z|>2$, then $(*)$ and $(\dagger)$ are valid and so, using the partial fraction decomposition $f(z)=\frac{1}{7}(\frac{3}{3z-1}-\frac{1}{z+2})$ we have

$$f(z)=\sum_{n=1}^{\infty}\left(\frac{3^{n-1}-(-2)^{n-1}}{7}\right)\frac{1}{z^n}.$$

The region of convergence for the above Laurent expansion is

$$D=\{z\in\mathbb{C}\mid |z|>2\}.$$

(b) The point $z=1$ is a point where the series $(**)$ and $(\dagger)$ converge. The common region of convergence of $(**)$ and $(\dagger)$ is $A=\{z\in\mathbb{C}\mid 1/3<|z|<2\}$. On A we have

$$f(z)=-\frac{1}{7}\sum_{n=1}^{\infty}(-2)^{n-1}z^{-n}-\frac{1}{7}\sum_{n=0}^{\infty}3^{n+1}z^n$$

The region of convergence of the above Laurent series is the annular region A.

(c) For the Maclaurin series for $f(z)$ to converge on a disc B_r, both summands of the partial fraction decomposition have to be analytic on B_r. The largest disc on which this happens when in the disc $B_{1/3}$. Here $(**)$ and $(\ddagger)$ are valid. Hence on $B_{1/3}$ we have

$$f(z)=\frac{1}{7}\sum_{n=0}^{\infty}\left(-3^{n+1}-\frac{(-1)^n}{2^{n+1}}\right)z^n.$$

As mentioned above, the region of convergence is the open disc $B_{1/3}$.

2. Evaluate $\displaystyle\int_0^{2\pi}\frac{d\theta}{1+a\cos\theta}$ where $0\le a<1$.

Solution: For such problems, note that on the unit circle C_1, centred at 0, we can write $z=e^{i\theta}$, and in this case

$$\cos\theta=\frac{1}{2}(z+1/z) \qquad \sin\theta=\frac{1}{2i}(z-1/z).$$

Moreover, again with $z=e^{i\theta}$, we have $z'(\theta)=ie^{i\theta}=iz(\theta)$. It follows that if we have an integral of the form $\int_0^{2\pi}f(\cos\theta,\sin\theta)d\theta$, then

$$\int_0^{2\pi}f(\cos\theta,\sin\theta)d\theta=\oint_{C_1}f\Big(\frac{z+1/z}{2},\frac{z-1/z}{2i}\Big)\frac{dz}{iz}.$$

In particlular,

$$\int_0^{2\pi}\frac{d\theta}{1+a\cos\theta}=\oint_{C_1}\frac{1}{1+\frac{a}{2}(z+\frac{1}{z})}\frac{dz}{iz}=\frac{2}{i}\oint_{C_1}\frac{dz}{az^2+2z+a}$$

Set

$$f(z)=\frac{2}{i}\cdot\frac{1}{az^2+2z+a}.$$

The integrand has poles at the zeros of $az^2 + 2z + a$, and the quadratic formula tells us that these occur at $z_1 = -\frac{1}{a} - \frac{\sqrt{1-a^2}}{a}$ and $z_2 = -\frac{1}{a} + \frac{\sqrt{1-a^2}}{a}$. An easy computation shows that $z_1 z_2 = \frac{1}{a^2} - \frac{1-a^2}{a^2} = 1$. Since $z_1 = -1-(\sqrt{1-a^2}/a)$, clearly z_1 is a real number less that -1 therefore $|z_1| > 1$. It follows that $|z_2| = 1/|z_1| < 1$. Thus $f(z)$ has only one isolated singularity in the interior of C_1, and that is a simple pole at z_2.

Let

$$g(z) = \frac{1}{a(z - z_1)}.$$

Then

$$f(z) = \frac{2}{i} \cdot \frac{g(z)}{z - z_2}.$$

From the Cauchy Integral Formula we get

$$\begin{aligned}\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} = \oint_{C_1} f(z)dz &= \frac{2(2\pi i)}{i} g(z_2) \\ &= 4\pi \frac{1}{a(z_2 - z_1)} \\ &= 4\pi \frac{a}{a(2\sqrt{1-a^2})} \\ &= \frac{2\pi}{\sqrt{1-a^2}}.\end{aligned}$$

3. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2 + 2x + 2)(x^2 + 1)} dx.$

Solution: Let R be a positive real number. Let S_R be the semicircle of radius R centred at 0, consisting of all points $z \in \mathbb{C}$ such that $|z| = R$ and and $\mathrm{Im}(z) \geq 0$. Orient S_R in the counterclockwise direction. Let $\Gamma_R = S_R + [-R, R]$, where $[-R, R]$ is the directed line segment on the real axis from $-R$ to R. Let

$$f(z) = \frac{1}{(z^2 + 2z + 2)(z^2 + 1)}.$$

The trick is to show that $\lim_{R\to\infty} \int_{S_R} f(z)dz = 0$. Then

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{[-R,R]} f(z)dz &= \lim_{R\to\infty} \int_{\Gamma_R} f(z)dz - \lim_{R\to\infty} \int_{S_R} f(z)dz \\ &= \lim_{R\to\infty} \int_{\Gamma_R} f(z)dz,\end{aligned}$$

and the last integral can be computed using contour integrals.

Now, factoring the denominator in the expression for $f(z)$, we see that

$$f(z) = \frac{1}{(z^2 + 2z + 2)(z^2 + 1)} = \frac{1}{(z - i)(z + i)(z + 1 - i)(z + 1 + i)}.$$

Clearly $f(z)$ has simple poles at $z = \pm i$ and $z = -1 \pm i$. If R is large ($R > \sqrt{2}$ will do), only two of these, namely $z = i$ and $z = -1 + i$ lie in the interior of Γ_R. The other two lie outside Γ_R. We have a partial fraction decomposition

$$f(z) = \frac{A}{z - i} + \frac{B}{z + i} + \frac{C}{z + 1 - i} + \frac{D}{z + 1 + i}.$$

Since $\frac{B}{z+i}$ and $\frac{D}{z+1+i}$ are analytic on and in the interior of Γ_R, their integrals over Γ_R is zero by the Cauchy Integral formula. Thus

$$\int_{\Gamma_R} f(z)dz = \int_{\Gamma_R} \frac{A}{z-i}dz + \int_{\Gamma_R} \frac{C}{z+1-i}dz = 2\pi i(A+C).$$

We have to find A and C. Standard method for partial fractions tell us that

$$A = \lim_{z\to i}(z-i)f(z) = \lim_{z\to i} \frac{1}{(z+i)(z+1-i)(z+1+i)} = \frac{1}{(2i)(1+2i)}$$

and

$$C = \lim_{z\to -1+i}(z+1-i)f(z) = \lim_{z\to -1+i} \frac{1}{(z-i)(z+i)(z+1+i)} = \frac{1}{(2i)(1-2i)}$$

Therefore

$$\int_{\Gamma_R} f(z)dz = (2\pi i)\frac{1}{2i}\left\{\frac{1}{1+2i} + \frac{1}{1-2i}\right\} = \frac{2\pi}{5}.$$

We have to show that $\lim_{R\to\infty} \int_{S_R} f(z)dz = 0$. On the semicircle S_R, $z = Re^{i\theta}$, for some $\theta \in [0, \pi]$, and hence

$$\lim_{R\to\infty} \frac{(z^2+2z+2)(z^2+1)}{R^4} = \lim_{R\to\infty} \frac{R^2e^{(2i\theta} + 2Re^{i\theta} + 2)(R^2e^{2i\theta}+1)}{R^4} = e^{4i\theta}.$$

Since $|e^{4i\theta}| = 1$, it follows that for large R, $|\frac{(z^2+2z+2)(z^2+1)}{R^4}| \geq \frac{1}{2}$. Thus for large R, $|f(z)| \leq \frac{2}{R^4}$. Hence, for large R,

$$\left|\int_{S_R} f(z)dz\right| \leq \frac{2}{R^4}\ell(S_R) = \frac{2\pi}{R^3} \longrightarrow 0 \quad \text{as } R \longrightarrow \infty.$$

Thus $\lim_{R\to\infty} \int_{S_R} f(z)dz = 0$. Now

$$\int_{-R}^{R} f(x)dx = \int_{\Gamma_R} f(z)dz - \int_{S_R} f(z)dz = \frac{2\pi}{5} - \int_{S_R} f(z)dz.$$

Letting $R \to \infty$ and using the fact that $\lim_{R\to\infty} \int_{S_R} f(z)dz = 0$, we get

$$\int_{-\infty}^{\infty} f(x)dx = \frac{2\pi}{5}.$$