LECTURE 20

Date of Lecture: March 29, 2022

1. Orders of zeros and poles

This section is not part of your second midterm.

As always, if $z_0 \in \mathbb{C}$ then we write $B_r(z_0)$ for the open disc of radius r centred at z_0.

1.1. Zeros of an analytic function.

Let f be analytic in a domain D and z_0 a point in D such that $f(z_0) = 0$. Let $B = B_r(z_0)$ be an open disc centred at z_0 such that $B \subset D$. Since D is an open set , such a disc B exists. Now f has a Taylor's series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where $a_n = f^{(n)}(z_0)/n!$. If f is not identically zero on B, we say f has a *isolated zero* at $z = z_0$. In this case there is an integer $k > 0$ such that $a_k \neq 0$ but $a_l = 0$ for $l < k$. Thus

$$f(z) = \sum_{n=k}^{\infty} a_n (z - z_0)^n = (z - z_0)^k \sum_{n=0}^{\infty} a_{n+k} (z - z_0)^n.$$

Let $g(z) = \sum_{n=0}^{\infty} a_{n_k} (z - z_0)^n$. Then g is analytic on B and $g(z_0) = a_k \neq 0$. Now on the punctured disc $B \smallsetminus \{z_0\}$, $g(z) = f(z)/(z - z_0)^k$, and hence $g(z)$ can be defined as an analytic function on D by setting it equal to $f(z)/(z - z_0)^k$ when $z \neq 0$.

The integer k is said to be the *order of the isolated zero of f at $z = z_0$*. Thus f has an isolated singularity at $z = z_0$ if and only if we can write

$$f(z) = (z - z_0)^k g(z)$$

with for some $k > 0$ such that $g(z_0) \neq 0$. If $a_0 \neq 0$ we often say that f has a zero of order $k = 0$ at z_0.

If f has an isolated zero or order one at z_0 (for example $f(z) = \sin z$, with $z_0 = 0$), then we often say f has a *simple zero* at z_0.

1.2. Isolated Singularities.

An analytic function f on a domain D is said to have an *isolated singularity* at $z_0 \in \mathbb{C}$ if $z_0 \notin D$, but there is an open disc $B = B_r(z_0)$ centred at z_0 such that the punctured disc $B \smallsetminus \{z_0\}$ is contained D. In other words z_0 is "surrounded" by D though it does not belong to D.

Suppose z_0 is an isolated singularity for an analytic function f. Let B be as above. Then f has a Laurent expansion

$$f(z) = \cdots + \frac{a_{-n}}{(z - z_0)^n} + \cdots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \ldots a_m (z - z_0)^m + \ldots$$

There are three possibilities, namely:

(i) The negatively indexed coefficients are all zero, i.e. $a_k = 0$ for all $k < 0$. In this case we say f has a *removable singularity at* z_0. This is because the power series represents an analytic function on B and so f can be extended to z_0 as an analytic function by defining $f(z_0) = a_0$, and in this way the singularity at z_0 can be "removed".

(ii) There is a positive integer n such that $a_{-n} \neq 0$ but $a_k = 0$ for all $k < -n$. In this case we say f has a *pole of order* n at $z = z_0$. A pole of order one is often called a *simple pole*.

(iii) There are an infinite number of negative integers k such that $a_k \neq 0$. In other words, the Laurent expansion of f has an infinite number of coefficients with negative indices. In this case we say the f has an *essential singularity at* $z = z_0$.

Examples 1.2.1. **(a)** $f(z) = (z^2 + 2iz - 1)e^z$ has an isolated zero of order 2 at $z = -i$.

(b) $f(z) = \dfrac{\sin z}{z^2 + 1}$ has simple zeros at πn, $n = 0, \pm 1, \pm 2, \ldots$, and simple poles at $z = \pm i$.

(c) $f(z) = e^{1/z}$ has an essential singularity at 0. In fact its Laurent series is obviously $f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{z^n}$.

1.3. **The Identity Principle.** Suppose U is the set of points w in D such that $f^{(n)}(w) = 0$ for all $n \geq 0$. Suppose U is non-empty. Let $z_0 \in U$. Then Taylor's series for f centred at z_0 is zero, since it is of the form $\sum_{n=0}^{\infty} (f^{(n)}(z_0))/n!(z - z_0)^n$. This means that f identically zero in every disc $B_r(z_0)$ centred at z_0 contained in D. Thus, by definition of an open set, U is open. On the other hand, if z^* is a limit point of U, then there exists a sequence points w_n in U converging to z^*. Since $f^{(n)}$ is differentiable, it is continuous, and hence $f^{(n)}(z^*) = \lim_{n\to\infty} f^{(n)}(w_n) = 0$. Thus $w^* \in U$. In other words U is also closed, which means $V = D \setminus U$ is open. Since $U \cup V = D$ and $U \cap V = \emptyset$ and D is connected, by an argument we gave earlier in the course,[1] either U or V is empty. We have assumed U is non-empty. So $V = \emptyset$, which means $U = D$. Thus f vanishes identically on D.

Suppose $z_0 \in D$ is such that $f(z_0) = 0$ but z_0 is not an isolated zero, i.e., in every circular neighbourhood of z_0 in D, there is a zero of f distinct from z_0. In that case all the coefficients a_n of the Taylor's expansion of f around z_0 are zero (otherwise, we can write $f(z) = (z - z_0)^k g(z)$ with $g(z_0) \neq 0$ by looking at the Taylor's expansion, and this means z_0 is an isolated zero of f, contradicting our assumption). From what we argued above, this means f is identically zero on D.

Thus if f is an analytic function on a domain which does not vanish identically, then the zeros of f are isolated.

Now suppose $f(z)$ is an entire function such that $f(x) = \sin(x)$ when x is a real number. Then the entire function $g(z) = f(z) - \sin(z)$ is identically zero on the real line. Thus g does not have isolated zeros on $\mathbb{R}$. Hence g is identically zero on $\mathbb{C}$. Hence $f(z) = \sin z$ for all $z \in \mathbb{C}$.

Similarly, we know that $\sin^2 x + \cos^x = 1$ for $x \in \mathbb{R}$. Let g be the constant function 1. Then $g(z) = \sin^2 z + \cos^z - 1$ is zero on the real axis, and hence its zeros there are not isolated. This in turn means that $g(z)$ is identically zero. Hence $\sin^2 z + \cos^2 z = 1$ for $z \in \mathbb{C}$.

[1] See Appendix to Lecture 17.

This phenomenon, that if two analytic functions on a domain agree on a line segment, or on any set with a limit point, then they must agree on their entire domain of definition is called the *identity principle*.

2. **Examples**

This section has material which is part of the midterm.

1. Let $f(z) = \dfrac{e^z}{e^z + 1}$.

(a) Does f have an antiderivative in the domain D shown in FIGURE 1 below?

(b) Let Γ be the semi-circle of radius 2 centred at 0 oriented in the *clockwise* direction. Evaluate $\int_\Gamma f(z)dz$.

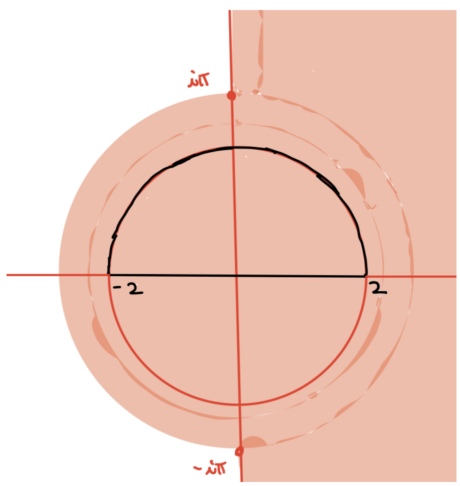

FIGURE 1. The domain D. The line segment starting from $i\pi$ and going "north" indefinitely is not included, nor is the line segment starting at $-i\pi$ and going south indeninitely. The points $i\pi$ and $-i\pi$ are not included either. The black semicircle is centred at 0 and of radius 2.

Solution: For part (a), note that $f(z)$ is analytic on D, since the zeros of the denominator e^z+1 occur only at $n(i\pi)$, with n an integer and these points are not in D. D is clearly simply connected. We know that an every analytic function on a simply connected domain has an antiderivative. So f has an anti-derivative.

For (b), since f is analytic on the simply connected domain, the given integral is the same as the integral of f along any contour in D starting at -2 and terminating at 2, e.g. the line segment $[-2,2]$. So

$$\begin{aligned}\int_\Gamma f(z)dz &= \int_{-2}^{2} f(x)dx \\ &= \int_{-2}^{2} \frac{e^x}{e^x+1}dx \\ &= \log\left(e^x+1\right)\Big]_{x=-2}^{x=2} \\ &= \log\left\{\frac{e^2+1}{e^{-2}+1}\right\} \\ &= \log e^2 \\ &= 2.\end{aligned}$$

2. Let $f(z) = \dfrac{z^5+4z^4+3z-1}{z^6}$. Evaluate $\oint_{|z|=1} f(z)dz$.
Solution: There are two ways of doing this problem. The first is a follows. Let $g(z) = z^5+4z^4+3z-1$. Then the integral is

$$\oint\limits_{|z|=1} \frac{g(z)}{z^6}dz = (2\pi i)\frac{g^{(5)}(0)}{5!} = (2\pi i)\frac{5!}{5!} = 2\pi i.$$

The second method is to note that for an integer n (positive or negative) z^n has an anti-derivative whenever $n \neq -1$, namely $z^{n+1}/(n+1)$. Now $f(z) = \frac{1}{z} + \frac{4}{z^2} + \frac{3}{z^5} - \frac{1}{z^6}$. Since all the summands in the sum, except the first term, have anti-derivatives, the integrals of these summands over loops in $\mathbb{C} \smallsetminus \{0\}$ is zero. Hence

$$\oint\limits_{|z|=1} f(z)dz = \oint\limits_{|z|=1} \frac{1}{z}dz + \oint\limits_{|z|=1} \left\{\frac{4}{z^2} + \frac{3}{z^5} - \frac{1}{z^6}\right\}dz = 2\pi i + 0 = 2\pi i.$$

3. Suppose f is entire and $|f(z)| \leq |e^{\sin z}|$ for $z \in \mathbb{C}$. What can you say about f? Give reasons.
Solution: One can show that $f(z) = ce^{\sin z}$ where c is a constant. Work out the reason yourself.

3. **Evaluating real integrals via contour integrals**

You will not be tested on the material from this section for your second midterm.

Example. Evaluate $\int_{-\infty}^{\infty} \dfrac{1}{(1+x^2)^2}dx$.

Solution: Extend the integrand to an analytic function f on $\mathbb{C} \smallsetminus \{\pm i\}$ by the formula

$$f(z) = \frac{1}{(1+z^2)^2}, \qquad (z \in \mathbb{C} \smallsetminus \{-i, i\}).$$

Let R be a real number with $R > 1$. Let A_R be the semi-circle centred at 0 of radius R directed from R to $-R$ as in FIGURE 2 below. Let $\Gamma_R = A_R + [-R, R]$.

The strategy is as follows. We will evaluate $\int_{\Gamma_R} f(z)dz = \int_{A_R} f(z)dz + \int_{-R}^{R} f(x)dx$ using the Cauchy Integral Formula, and then show that $\lim_{R\to\infty} \int_{A_R} f(z)dz = 0$.

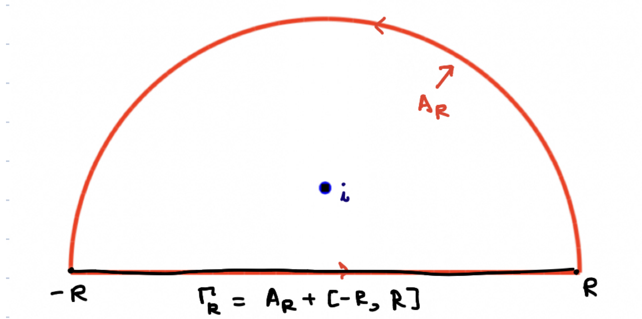

FIGURE 2.

Since $\lim_{R\to\infty} \int_{-R}^{R} f(x)dx = \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2}dx$, we will be able to arrive at the answer.

Let us first evaluate $\int_{\Gamma_R} f(z)dz$. We have, with $g(z) = (z+i)^{-2}$,

$$(\dagger)\qquad \begin{aligned} \int_{\Gamma_R} f(z)dz = \int_{\Gamma_R} \frac{1}{(z+i)^2(z-i)^2}dz &= \int_{\Gamma_R} \frac{g(z)}{(z-i)^2}dz \\ &= (2\pi i)g'(i) \\ &= (2\pi i)(-2)(z+i)^{-3}\Big|_{z=i} \\ &= (2\pi i)(-2)\frac{1}{(-8i)} \\ &= \frac{\pi}{2}. \end{aligned}$$

Now, from FIGURE 2,

$$(\ddagger)\qquad \int_{\Gamma_R} f(z)dz = \int_{A_R} f(z)dz + \int_{-R}^{R} f(x)dx.$$

We will now show that $\lim_{R\to\infty} \int_{A_R} f(z)dz = 0$. If z lies on A_R, then $z = Re^{it}$ for some R. In this case $1 + z^2 = 1 + R^2e^{2it}$. Since $|a-b| \geq |a| - |b|$, we have $|1+z^2| = |1+R^2e^{2it}| \geq R^2 - 1$. We therefore have,

$$\left|\int_{A_R} f(z)dz\right| \leq \frac{1}{(R^2-1)^2}\ell(A_R) = \frac{R\pi}{(R^2-1)^2}.$$

Now $\lim_{R\to\infty} \frac{R\pi}{(R^2-1)^2} = 0$. It follows that $\lim_{R\to\infty} \int_{A_R} f(z)dz = 0$. From this and from the equations ($\dagger$) and ($\ddagger$) above, we get,

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx = \lim_{R\to\infty} \int_{\Gamma_R} f(z)dz - \lim_{R\to\infty} \int_{A_R} f(z)dz = \frac{\pi}{2}.$$

As an exercise, evaluate the same integral using the substitution $x = \tan\theta$ and the formula $\cos^2\theta = \frac{1}{2}(\cos(2\theta) + 1)$.