Algebraic operations:

Let $z_1, z_2, z_3 \in \mathbb{C}$.

- $z_1 + z_2 = z_2 + z_1$
- $z_1 z_2 = z_2 z_1$
- $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$.

Let us prove the third property.

Let $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$, $z_3 = a_3 + ib_3$, $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$.

$$
\begin{aligned}
&(a_1 + ib_1)\{(a_2 + ib_2) + (a_3 + ib_3)\} \\
&= (a_1 + ib_1)\{(a_2 + a_3) + i(b_2 + b_3)\} \\
&= a_1(a_2 + a_3) - b_1(b_2 + b_3) + i\{a_1(b_2 + b_3) + b_1(a_2 + a_3)\} \\
&= a_1 a_2 + a_1 a_3 - b_1 b_2 - b_1 b_3 \\
&\qquad + i\{a_1 b_2 + a_1 b_3 + b_1 a_2 + b_1 a_3\} \\
&= a_1 a_2 - b_1 b_2 + i\{a_1 b_2 + b_1 a_2\} \\
&\quad + a_1 a_3 - b_1 b_3 + i\{a_1 b_3 + b_1 a_3\} \\
&= z_1 z_2 + z_1 z_3.
\end{aligned}
$$

So $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$.

Complex conjugates (again):

Recall that if $z = a + ib \in \mathbb{C}$, $Re(z) = a$, $Im(z) = b$, then its complex conjugate is

$$\boxed{\bar{z} = a - ib.}$$

Last time, we saw that

$$z\bar{z} = (a+ib)(a-ib) = a^2+b^2 = |z|^2.$$

So

① $$\boxed{z\bar{z} = |z|^2}$$

If $z \neq 0$, then

② $$\frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

Proof

$$\frac{1}{z} = \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}.$$

③ $$\boxed{|z_1 z_2| = |z_1| \cdot |z_2|}$$

Proof of ③: Enough to show

$$|z_1 z_2|^2 = |z_1|^2 \cdot |z_2|^2$$

The left side $= (z_1 z_2)\overline{(z_1 z_2)}$ (check $\overline{z_1 z_2} = \bar{z_1}\,\bar{z_2}$)

$= z_1 z_2 (\bar{z_1}\,\bar{z_2})$

$= z_1 \bar{z_1}\; z_2 \bar{z_2}$

$= |z_1|^2\, |z_2|^2$

Examples

1. Calculate $|(1+2i)^{10}|$.

Soln: From the above formulas

$$|(1+2i)^{10}| = |1+2i|^{10}$$

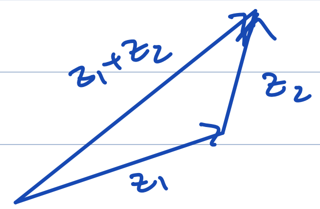
$$= \left(\sqrt{1^2+2^2}\right)^{10}$$

$$= (\sqrt{5})^{10}$$
$$= 5^5 \longleftarrow \text{Ans.}$$

Triangle inequality:

Using the vector interpretation of complex numbers:

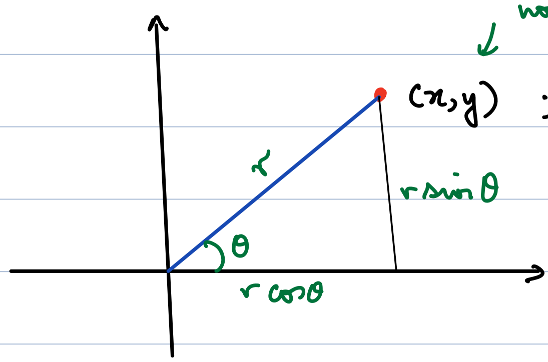
(a) $|z_1 + z_2| \le |z_1| + |z_2|$ (Δ-ineq)



(b) $|z_1 - z_2| \ge \big| |z_1| - |z_2| \big|$ (Exercise)

Hint: Let $w_1 = z_1 - z_2$, $w_2 = z_2$.
Use the Δ- inequality on $w_1 + w_2$

Polar form of a complex:



$x = r\cos\theta$
$y = r\sin\theta$

It follows that if $z \in \mathbb{C}$, $\mathrm{Re}(z) = x$, $\mathrm{Im}(z) = y$, then, if $z \neq 0$,

$$\boxed{z = r(\cos\theta + i\sin\theta)}$$ The polar form of $z \neq 0$.

The angle θ is ambiguous. We know it up to addition by $2\pi k$, where k is an integer.

$\mathbb{Z}$ = set of integers.

$r = |z|$. No ambiguity.

θ is called an argument of z.

It is ambiguous: it is a "multi-valued" function

Pick any one argument θ_0 of z.

$$\arg(z) = \{ \theta_0 + 2\pi k \mid k \in \mathbb{Z} \}.$$

Can we make things less ambiguous?

Pick any number $\tau \in \mathbb{R}$. Then

$$\arg(z) \cap (\tau, \tau + 2\pi]$$

consists of only one point. This is single element is denoted $\arg_\tau(z)$.

The Principal Argument:

Let $z \in \mathbb{C} \setminus \{0\}$. The principal argument of z is

$$\operatorname{Arg}(z) = \arg_{-\pi}(z).$$

In other words $\operatorname{Arg}(z)$ is the only argument of z lying in $(-\pi, \pi]$.

Examples:

1. If $r = 2$, $\theta = \pi/3$, what is z?

Ans: $z = 2(\cos(\pi/3) + i\sin(\pi/3))$

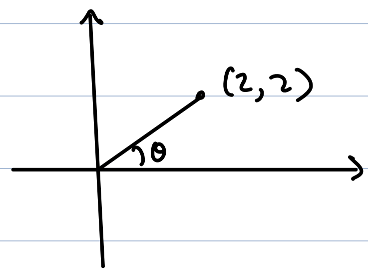
$$= 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

$$= 1 + i\sqrt{3}.$$

2. If $z = 2+2i$, write a polar form of z.

<u>Soln</u>:

$$r = |z| = \sqrt{2^2+2^2} = (\sqrt{2})(2).$$



$$\tan\theta = \frac{2}{2} = 1.$$

$$\theta = \tan^{-1}(1) = \pi/4.$$

$$z = 2\sqrt{2}\left(\cos(\pi/4) + i\sin(\pi/4)\right)$$

<u>Question</u>: Is $r = |z|$, and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ a good formula for the polar form?

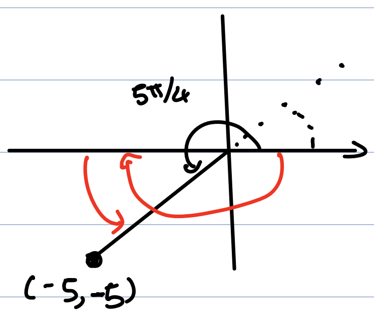
<u>Ans</u>: No. Take $z = -5-5i$.

$r = 5\sqrt{2}$ (check this).

$$\tan^{-1}(y/x) = \tan^{-1}\left(\frac{-5}{-5}\right) = \tan^{-1}(1)$$

$= \pi/4$

Not an argument of $-5-5i$



$5\pi/4$ is an argument.

$$\arg(-5-5i) = \{5\pi/4 + 2k\pi \mid k \in \mathbb{Z}\}$$

Note that in the above the only member of the set $\arg(-5-5i)$ lying in $(-\pi, \pi]$ is $-\frac{3\pi}{4}$.

So $\operatorname{Arg}(-5-5i) = -\frac{3\pi}{4}$.

The complex exponential:

We want to make sense of e^z, when $z \in \mathbb{C}$.
It should have the following properties:

(a) $e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$, $e^0 = 1$

(b) $\frac{d e^{iy}}{dy} = i e^{iy}$, $y \in \mathbb{R}$.

Let $z = x + iy$. Then, if we have an exponential function with the above properties,

$$e^z = e^{x+iy} = e^x \cdot e^{iy}$$

have already defined this.

It remains to define e^{iy}.
It has to satisfy the following:—

$$\frac{d e^{iy}}{dy} = i e^{iy}$$

$$\frac{d^2 e^{iy}}{dy^2} = i(i e^{iy}) = -e^{iy} \quad (\text{since } i^2 = -1)$$

Recall from diff. eqns, if $g(y)$ is a function such that $\frac{d^2}{dy^2} g(y) = -g(y)$ then

$$g(y) = c_1 \cos(y) + c_2 \sin(y)$$

The above is from the theory of diff. eqns.

So $\quad e^{iy} = c_1 \cos(y) + c_2 \sin(y).$ —— ①

Also, since $\frac{d}{dy} e^{iy} = i e^{iy}$, we get

$$i e^{iy} = c_2 \cos(y) - c_1 \sin(y). \quad \text{—— ②}$$

Set $y=0$ in ①. Get

$$1 = c_1.$$

Set $y=0$ in ②. Get

$$i = c_2.$$

So $\quad e^{iy} = \cos y + i \sin y.$

Definition: $\quad e^{x+iy} = e^x (\cos y + i \sin y), \quad x, y \in \mathbb{R}.$

Recall that

$$\cos(\theta_1+\theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$$

$$\sin(\theta_1+\theta_2) = \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2.$$

Using this, one checks easily that

$$e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$$

$$(\cos\theta_1 + i \sin\theta_1) \cdot (\cos\theta_2 + i \sin\theta_2)$$

$$= (\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i(\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2)$$

$$= \cos(\theta_1+\theta_2) + i \sin(\theta_1+\theta_2).$$

thus $e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$

This also shows that

$$e^{(x_1+iy_1)} \cdot e^{(x_2+iy_2)}$$

$$= e^{x_1+x_2+i(y_1+y_2)}$$

So $e^{z_1} \cdot e^{z_2} = e^{z_1+z_2}$.

$$\frac{(\cos\theta_1 + i\sin\theta_1)}{\cos\theta_2 + i\sin\theta_2} = \frac{e^{i\theta_1}}{e^{i\theta_2}} \overset{\text{check}}{=} e^{i(\theta_1-\theta_2)}$$

$$= \cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2).$$

The polar form now becomes (for $z \neq 0$)

$$z = r e^{i\theta}$$

where $r = |z|$, and $\theta \in \arg(z)$.

Note that

$$e^{i(\theta+2\pi k)} = e^{i\theta}.$$

So $e^{i\theta}$ is 2π-periodic.

Suppose $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$ (polar forms)

then

$$\boxed{\begin{aligned} z_1 z_2 &= r_1 r_2 e^{i(\theta_1+\theta_2)} \\ \frac{z_1}{z_2} &= \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)} \end{aligned}}$$

$$\boxed{\begin{aligned} z_1 z_2 &= r_1 r_2\, e^{i(\theta_1+\theta_2)} \\ \frac{z_1}{z_2} &= \frac{r_1}{r_2}\, e^{i(\theta_1-\theta_2)} \end{aligned}}$$

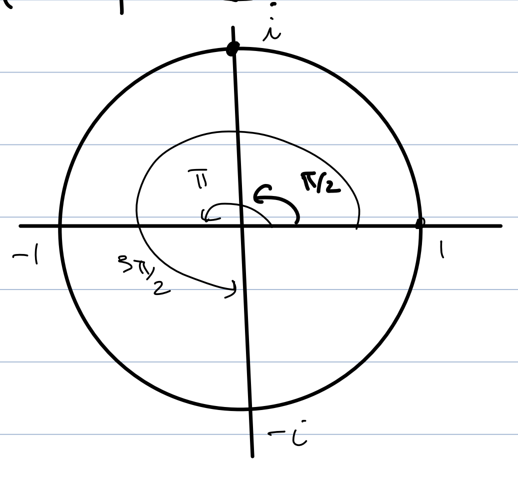
Note : $\theta \in \mathbb{R}$

(1) $e^{i0} = 1$, $e^{i\pi/2} = i$, $e^{i\pi} = -1$, $e^{3\pi/2} = -i$.

We are using the fact that

$$|\cos\theta + i\sin\theta| = \sqrt{\cos^2\theta + \sin^2\theta} = 1.$$

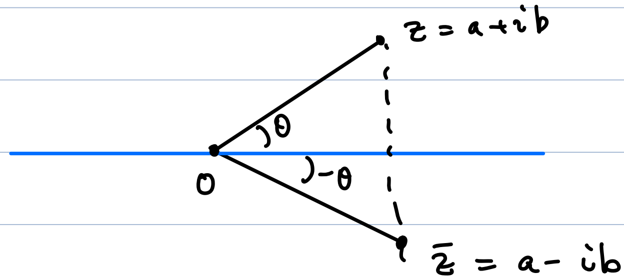
So $|e^{i\theta}| = 1$.



(2) $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$, $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

We are using : $\cos(-\theta) = \cos\theta$
$\sin(-\theta) = -\sin\theta$.

(3) $z = re^{i\theta}$, then $\bar{z} = re^{-i\theta}$.



Example : 1. $\dfrac{e^{1+3\pi i}}{e^{-1+3\pi i}} = ?$ (do this yourself)

2. Prove that $|e^z| \leq 1$ iff $Re(z) \leq 0$

Pf : Suppose $z = x + iy$, $x, y \in \mathbb{R}$.

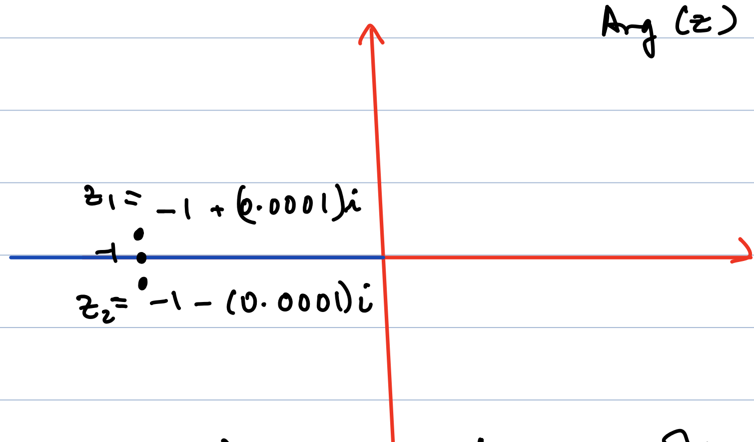
$e^z = e^x \cdot e^{iy}$.

so $|e^z| = e^x$ (since $|e^{iy}| = 1$)

The right side is ≤ 1 if and only if $x \leq 0$. //

Branch cut for Arg(z)

$Arg(z) \in (-\pi, \pi]$



Then $Arg(z_1) \approx \pi$, $Arg(z_2) \approx -\pi$.

So $\mathrm{Arg}(z)$ has a discontinuity along the negative x-axis. This is called a <u>branch cut</u>.

In general $\arg_\tau(z)$ has a discontinuity along the line making an angle τ with the x-axis.

<u>Roots of unity</u> : What are the n^{th} roots of 1?

$$1 = e^{i 2\pi k}, \quad k \in \mathbb{Z}.$$

Note that $\left(e^{i(2\pi k)/n}\right)^n = e^{i(2\pi k)} = 1$

So $e^{i(2\pi k)/n}$ is an n^{th} root of 1.

<u>Check</u> : $\underset{e^{i(0)/n}}{\overset{1}{\shortparallel}}, e^{i(2\pi)/n}, e^{i(4\pi)/n}, \dots, e^{i(2\pi(n-1))/n}$

are the distinct n^{th} roots of 1.

These are the solutions of

$$z^n - 1 = 0$$