LECTURE 19

Date of Lecture: March 24, 2022

Many thanks to Diana Ryoo for lending me her copy of the class notes.

1. Examples of Taylor and Maclaurin series

1.1. **Recap of basic facts about power series.** From the last lecture we know that if f is analytic in $B_r(z_0)$, the disc of radius r centred at z_0, then it has a power series expansion, called the Taylor series for f at z_0

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n, \qquad z \in B_r(z_0). \tag{1.1.1}$$

If $z_0 = 0$, the series in (1.1.1) is called the Maclaurin series for f.

We also know that a power series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

has a radius of convergence R such that for $|z - z_0| < R$ the power series converges, and for $|z - z_0| > R$ the series diverges. In the appendix for this lecture we will prove that if $\lim_{n\to\infty} |a_n|^{1/n}$ exists, then a power series can be computed by the following formula

$$R = \frac{1}{\lim\limits_{n\to\infty} |a_n|^{1/n}}. \tag{1.1.2}$$

We will prove (1.1.2) in the appendix (see Theorem A.1). We proved in the last class that within the radius of convergence of $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, the function

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is analytic, with derivative

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}.$$

The above formula also gives us,

$$a_n = \frac{f^{(n)}(z_0)}{n!}. \tag{1.1.3}$$

1.2. **Examples.** Here are some examples.

1. Let

$$f(z) = e^z.$$

Find the Maclaurin's series for f.

Solution: Since f is entire, the corresponding Maclaurin series is convergent for all $z \in \mathbb{C}$. Now $f^{(n)}(z) = e^z$ for all $n \geq 0$ and hence $f^{(n)}(0) = 1$ for all $n \geq 0$. Thus the Maclaurin series for e^z is

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

2. Let

$$f(z) = \sin z.$$

Find the Maclaurin's series for f.

Solution: Once again we have an entire function, and so the Maclaurin series will converge for all complex numbers z. Since $f'(z) = \cos z$, and $f''(z) = -\sin z$, one sees easily that $f^{(2n)}(z) = (-1)^n \sin z$ and $f^{(2n+1)}(z) = (-1)^n \cos z$, for $n \geq 0$. This means $f^{(2n)}(0) = 0$, and $f^{(2n+1)}(0) = (-1)^n$. It is now easy to see that

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \frac{1}{7!}z^7 + \dots.$$

3. Let

$$f(z) = \cos z.$$

Find the Maclaurin's series for f.

Solution: Using the same technique as above (note that $f^{(2n)}(z) = (-1)^n \cos z$ and and $f^{(2n+1)}(z) = (-1)^{n+1} \sin z$, we see that

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 - \frac{1}{6!}z^6 + \dots.$$

Remark 1.2.1. We do not have to use (1.1.3) to find the Maclaurin series for $\sin z$ and $\cos z$. One can also work these out by using the formulas

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \text{and} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

and then substituting the Maclaurin series for e^{iz} and e^{-iz} from 1.

4. Let

$$f(z) = 1/(1 - z).$$

Find the Maclaurin's series for f. What is the radius of convergence R of this Maclaurin series?

Solution: Then $f^{(n)}(z) = n!/(1-z)^{n+1}$, $n \geq 0$. Thus $f^{(n)}(0)/n! = 1$, $n \geq 0$. The Maclaurin series for f is therefore, $\sum_{n=0}^{\infty} z^n$. The radius of convergence is

$$R = \frac{1}{\lim\limits_{n\to\infty} |a_n|^{1/n}},$$

where $a_n = 1$ for every n. Thus $R = 1$. Thus

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots + z^n + \ldots$$

for $|z| < 1$.

1.3. **Leibniz's rule and the Cauchy product of power series.** Suppose f and g are analytic on a domain D. A repeated application of the product rule gives the formula

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}. \tag{1.3.1}$$

The formula (1.3.1) is called the *Leibniz rule*. Now suppose $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ and $\sum_{n=0}^{\infty} b_n(z-z_0)^n$ are power series, with radius of convergence R_1 and R_2 respectively. Let r be a positive number less than the minimum of R_1 and R_2. On $B_r(z_0)$ we have two analytic functions, namely $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ and $g(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^n$, Applying Leibniz's rule above as well as the formula (1.1.3) we see that on $B_r(z_0)$ we have

$$\Big(\sum_{n=0}^{\infty} a_n(z-z_0)^n\Big)\Big(\sum_{n=0}^{\infty} b_n(z-z_0)^n\Big) = \sum_{n=0}^{\infty} c_n(z-z_0)^n, \tag{1.3.2}$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}. \tag{1.3.3}$$

5. Find the first three nonzero terms of the Maclaurin series for

$$f(z) = e^z \sin z$$

using the Cauchy product of power series.

Solution: Let $e^z \sin z = \sum_{n=0} c_n z^n$. Then

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0$$

where a_n are the coefficients of the Maclaurin series for e^z and b_n the coefficients of the Maclaurin series for $\sin z$. Note that $a_n = 1/n!$ for all $n \geq 0$ and $b_0 = 0$, $b_1 = 1$, $b_2 = 0$, $b_3 = -(1/3!)$, $b_4 = 0$, $b_5 = 1/5!$, $b_6 = 0$, $b_7 = -(1/7!)$. Then

$c_0 = a_0 b_0 = (1)(0) = 0$

$c_1 = a_0 b_1 + a_1 b_0 = (1)(1) + (1)(0) = 1$

$c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0 = (1)(0) + (1)(1) + (1/2)(0) = 1$

$c_3 = a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0 = (1)(-1/6) + (1)(0) + (1/2)(1) + (1/6)(0) = 1/3$

Thus

$$e^z \sin z = z + z^2 + \tfrac{1}{3} z^3 + \ldots$$

6. Let

$$f(z) = \frac{1}{1 + \csc z}$$

where $\csc z = 1/\sin z$. Find the first three nonzero terms of the Maclaurin series for f.

Solution: A simple computation shows that

$$f(z) = \frac{\sin z}{1 + \sin z}.$$

We will use a technique called the *method of undetermined coefficients*. First note that

$$f(z)(1 + \sin z) = \sin z.$$

Let $f(z) = \sum_{n \geq 0} a_n z^n$ and $1 + \sin z = \sum_{n \geq 0} b_n z^n$. Then $\sin z = \sum_{n \geq 0} c_n z^n$, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. Since we know b_n and c_n, we should be able to solve for a_n. We have

$$\begin{aligned} b_0 = 1,\, b_1 = 1,\, b_2 = 0,\, b_3 = -\frac{1}{6},\, b_4 = 0,\, b_5 = \frac{1}{120} \\ c_0 = 0,\, c_1 = 1,\, c_2 = 0,\, c_3 = -\frac{1}{6},\, c_4 = 0,\, c_5 = \frac{1}{120} \end{aligned}$$

The equations we have to solve are

$$\begin{aligned} &(a_0)(1) = 0 \\ &(a_0)(1) + (a_1)(1) = 1 \\ &(a_0)(0) + (a_1)(1) + (a_2)(1) = 0 \\ &(a_0)(-1/6) + (a_1)(0) + (a_2)(1) + (a_3)(1) = -\frac{1}{6} \\ &\vdots \end{aligned}$$

From this we see that $a_0 = 0$, $a_1 = 1$, $a_2 = -1$, $a_3 = \frac{5}{6}$. The Maclaurin series for $f(z)$, with the first three nonzero terms being displayed, is

$$f(z) = z - z^2 + \tfrac{5}{6} z^3 + \ldots$$

7. Let $f(z) = \sum_{n=0}^{\infty} \frac{n^4}{2^n} z^n$. Compute

(a) The radius of convergence of f

(b) $f^{(5)}(0)$

(c) $\oint\limits_{|z|=1} f(z) dz$

(d) $\oint\limits_{|z|=1} \frac{f(z) \cos z}{z^2} dz$

Solution: Let $a_n = \frac{n^4}{2^n}$. Note that $|a_n| = a_n$.

(a) Using logarithms for positive real numbers and L'Hôpital's Rule, we see that $\lim_{n\to\infty} (n^4)^{1/n} = 1$. It follows that $\lim_{n\to\infty} a_n^{1/n} = 1/2$. Thus the radius of convergence is $R = 2$.

(b) $f^{(5)}(0) = (3!)a_5 = (5!)\dfrac{5^4}{2^5}$.

(c) Since the radius of convergenceof the given power seris is 2, f is analytic in $D = B_2(0)$, the open disc of radius 2 centred at $z = 0$. The circle $|z| = 1$ is contained in D. By the Cauchy Integral Theorem

$$\oint_{|z|=1} f(z)dz = 0.$$

(d) Let $g(z) = f(z)\cos z$. Since $f(z)$ is analytic on $B_2(0)$ and $\cos z$ is entire, $g(z)$ is analytic on $B_2(0)$. Thus

$$\begin{aligned}
\oint_{|z|=1} \frac{f(z)\cos z}{z^2} dz &= \oint_{|z|=1} \frac{g(z)}{z^2} dz \\
&= (2\pi i) g'(0) \\
&= (2\pi i)(f'(0)\cos(0) - f(0)\sin(0)) \\
&= (2\pi i) f'(0) = (2\pi i) a_1 \\
&= (2\pi i)(1/2) = \pi i.
\end{aligned}$$

2. **Laurent Series**

2.1. **Analytic functions on annuli.** The region between to concentric circles is called an *annulus*. When we include the the two bounding circles in the region is is called a *closed annulus* and when we exclude both of them, it is called an *open annulus*. The smaller circle's radius is usually denoted r, and the larger circle's radius as R. If the common centre is z_0 then the open annulus we are discussing is the set

$$\{z \in \mathbb{C} \mid r < |z - z_0| < R\},$$

and the closed annulus is

$$\{z \in \mathbb{C} \mid r \le |z - z_0| \le R\}.$$

There are half-open annuli, in which one of the boundary circles is included and the other excluded.

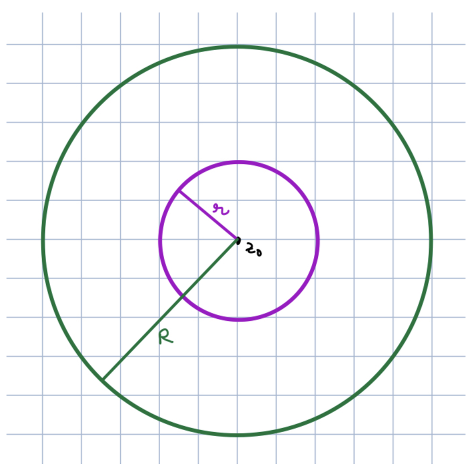

FIGURE 1. The region between the purple circle and the green circle is an annulus centred at z_0

Here is the main theorem (which we will prove after doing a long example).

Theorem 2.1.1. *Let f be analytic on the annulus $\{z \in \mathbb{C} \mid r < |z - z_0| < R\}$. Then f can be expressed on this annulus as the sum of two series*

$$f(z) = \sum_{n=1}^{\infty} \frac{a_{-n}}{(z - z_0)^n} + \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

The coefficients a_n, for n an integer (positive or negative), are given by

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

where C is a positively oriented simple loop containing in the annulus with z_0 in its interior.

The series in the theorem is called the *Laurent expansion* or the *Laurent series* of f in the annulus $\{z \in \mathbb{C} \mid r < |z - z_0| < R\}$.

One rarely uses the integral formula to compute the coefficients as the following example shows.

8. Find the Laurent expansion of

$$f(z) = \frac{z}{(z-1)(z-3)}$$

in

(a) the annulus $A = \{z \in \mathbb{C} \mid 1 < |z| < 3\}$;
(b) the disc $B_1(0) = \{z \in \mathbb{C} \mid |z| < 1\}$; and
(c) the open set $D = \{z \in \mathbb{C} \mid |z| > 3\}$.

Solution: We will repeatedly use the power series in Example 4, namely

$$(*) \qquad \frac{1}{1-w} = 1 + w + w^2 + \cdots + w^n + \ldots \qquad (|w| < 1).$$

We have the partial fraction decompostion

$$\frac{z}{(z-1)(z-3)} = \frac{-(1/2)}{z-1} + \frac{3/2}{z-3}.$$

(a) Let $w = 1/z$. On A, $|z| > 1$, whence $|w| < 1$ and so $(*)$ applies to w. Now

$$\begin{aligned} \frac{1}{z-1} = \frac{w}{1-w} &= w \sum_{n=0}^{\infty} w^n \qquad \text{(from } (*)) \\ &= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \\ &= \sum_{n=1}^{\infty} \frac{1}{z^n}. \end{aligned}$$

Next, since $|z| < 3$ on A, therefore $|z/3| < 1$ and hence $(*)$, applies to $w = z/3$. Thus

$$\begin{aligned}\frac{1}{z-3} &= -\left(\frac{1}{3}\right)\frac{1}{1-(z/3)} \\ &= -\frac{1}{3}\sum_{n=0}^{\infty}\frac{z^n}{3^n} \qquad \text{(from } (*)) \\ &= -\sum_{n=0}^{\infty}\frac{z^n}{3^{n+1}}.\end{aligned}$$

Combining our calculations we get

$$\frac{z}{(z-1)(z-3)} = \sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)z^n - \sum_{n=0}^{\infty}\left(-\frac{1}{(2)3^n}\right)z^n.$$

as the Laurent expansion of f in A.

(b) Since $B_1(0)$ is a disc, we expect that the Laurent series will be a power series, i.e. we expect $a_n = 0$ for $n < 0$. We have already seen in part **(a)** that

$$\frac{1}{z-3} = -\sum_{n=0}^{\infty}\frac{z^n}{3^{n+1}}$$

whenever $|z| < 3$, and in this is certainly true for $z \in B_1(0)$. Also, since $|z| < 1$ in $B_1(0)$, the identity $(*)$ gives us

$$\frac{1}{z-1} = -\frac{1}{1-z} = -\sum_{n=0}^{\infty}z^n$$

for $z \in B_1(0)$. Using our partial fraction decomposition $z/((z-1)(z-3)) = -(1/2)/(z-1) + (3/2)/(z-3)$, we see that

$$f(z) = \frac{1}{2}\sum_{n=0}^{\infty}\left(1 - \frac{1}{3^n}\right)z^n$$

for $z \in B_1(0)$.

(c) In part **(a)** we saw that

$$\frac{1}{z-1} = \sum_{n=1}^{\infty}\frac{1}{z^n}$$

whenever $|z| > 1$, and on $D = \{z \mid |z| > 3\}$, this condition is certainly fulfilled, and so the above expansion is valid in D.

Next note that $|z| > 3$ in D, if we set $w = 3/z$ then for $z \in D$, $|w| < 1$ and hence $(*)$ applies to w. Thus

$$\begin{aligned}\frac{1}{z-3} = \frac{1}{z}\cdot\frac{1}{1-w} &= \frac{1}{z}\sum_{n=0}^{\infty}w^n \qquad \text{(from } (*)) \\ &= \sum_{n=0}^{\infty}\frac{3^n}{z^{n+1}} \\ &= \sum_{n=1}^{\infty}\frac{3^{n-1}}{z^n}.\end{aligned}$$

Now using the partial fraction decomposition of $z/((z-1)(z-3))$ we get

$$\frac{z}{(z-1)(z-2)} = \sum_{n=1}^{\infty}\Big(\frac{3^n-1}{2}\Big)z^{-n} \qquad (|z|>3).$$

2.2. **Proof of Theorem 2.1.1.** Let A be the annulus depicted in Figure 1, i.e.

$$A = \{z \in \mathbb{C} \mid r < |z-z_0| < R\}.$$

Let C_r be the circle of radius r centred at z_0 and C_R the circle of radius R centred at z_0. Orient both of them positively. We will show (soon) that the following formula is true (for $z \in A$):

$$f(z) = \frac{1}{2\pi i}\oint_{C_R}\frac{f(\zeta)}{\zeta - z}d\zeta - \frac{1}{2\pi i}\oint_{C_r}\frac{f(\zeta)}{\zeta - z}d\zeta \qquad (z \in A). \tag{2.2.1}$$

Assume (2.2.1) is true. If $\zeta \in C_R$, and $z \in A$, then $|z-z_0| < R = |\zeta - z_0|$. This means $|(z-z_0)/(\zeta-z_0)| < 1$. Therefore

$$\begin{aligned}\frac{1}{\zeta - z} &= \frac{1}{(\zeta-z_0)-(z-z_0)}\\ &= \frac{1}{\zeta - z_0}\cdot\frac{1}{1-(z-z_0)/(\zeta-z_0)}\\ &= \sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(\zeta-z_0)^{n+1}} \qquad (\text{for } \zeta \in C_R \text{ and } z \in A)\end{aligned} \tag{2.2.2}$$

since $|(z-z_0)/(\zeta-z_0)| < 1$ for $\zeta \in C_R$ and $z \in A$.

If on the other hand $\zeta \in C_r$ and $z \in A$, then $|z-z_0| > r = |\zeta - z_0|$, which means $|(\zeta-z_0)/(z-z_0)| < 1$. In this case we have

$$\begin{aligned}\frac{1}{\zeta - z} &= \frac{1}{(\zeta-z_0)-(z-z_0)}\\ &= -\frac{1}{z - z_0}\cdot\frac{1}{1-(\zeta-z_0)/(z-z_0)}\\ &= -\sum_{n=0}^{\infty}\frac{(\zeta-z_0)^n}{(z-z_0)^{n+1}} \qquad (\text{for } \zeta \in C_r \text{ and } z \in A).\end{aligned} \tag{2.2.3}$$

Arguing as we did in Lecture 18, one can show that for $z \in A$,

$$\begin{aligned}\frac{1}{2\pi i}\oint_{C_R}\frac{f(\zeta)}{\zeta - z}d\zeta &= \frac{1}{2\pi i}\oint_{C_R}\Big(\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(\zeta-z_0)^{n+1}}\Big)f(\zeta)d\zeta \qquad (\text{from } (2.2.2))\\ &= \sum_{n=0}^{\infty}\left\{\frac{1}{2\pi i}\oint_{C_R}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}d\zeta\right\}(z-z_0)^n\end{aligned} \tag{2.2.4}$$

We have interchanged the sum $\sum_{n=0}^{\infty}$ with the integral $\oint_{C_R}$ and this needs justification, and this is where the arguments from Lecture 18 for power series apply. We do not want to revisit that, but for those interested in the technical issues, the basic idea is that one chooses a smaller circle of radius r_1 such that $|z| < r_1 < R$, and sets $\varrho = r_1/R$. Then $\varrho < 1$ and hence for every $\epsilon > 0$, there exists a number $K \geq 0$ such that $\sum_{n=k+1}^{\infty}\varrho^n < \epsilon$ for all $k \geq K$. It then follows easily that

$\sum_{n=k+1}^{\infty}|f(\zeta)|/|\zeta - z_0|^{n+1} \leq M\epsilon/(R - r_1)$ for all $k \geq K$ where M is the maximum of $f(\zeta)$ on C_R.

Similarly, and once again omitting technical details involving the difficulties with setting $\sum_{n=0}^{\infty} \oint = \oint \sum_{n=0}^{\infty}$ (this time one has to choose a circle C of radius r_1 such that $r < r_1 < |z|$ and set $\varrho = r/r_1$), one sees, using (2.2.3) that

$$\begin{aligned}\frac{1}{2\pi i}\oint_{C_r}\frac{f(\zeta)}{\zeta - z}d\zeta &= \sum_{n=0}^{\infty}\left\{\frac{1}{2\pi i}\oint_{C_r} f(\zeta)(\zeta - z_0)^n d\zeta\right\}(z - z_0)^{-n-1} \\ &= \sum_{m=-\infty}^{-1}\left\{\frac{1}{2\pi i}\oint_{C_r} f(\zeta)(\zeta - z_0)^{-m-1}\right\}(z - z_0)^m\end{aligned} \tag{2.2.5}$$

Finally if C is any simple loop in A with z_0 in the interior, then C_R and C_r can both be deformed in A to C, whence

$$\frac{1}{2\pi i}\oint_{C_R}\frac{f(\zeta)}{(\zeta - z_0)^{n+1}}d\zeta = \frac{1}{2\pi i}\oint_{C}\frac{f(\zeta)}{(\zeta - z_0)^{n+1}}d\zeta = \frac{1}{2\pi i}\oint_{C_r}\frac{f(\zeta)}{(\zeta - z_0)^{n+1}}d\zeta$$

for all integers (positive or negative) n, since $f(z)/(z - z_0)^{n+1}$ is analytic on A. From (2.2.4) and (2.2.5), we get the theorem—provided we prove (2.2.1).

We now prove (2.2.1). Pick a small circle C around z lying wholly in A, as in the right side of FIGURE 2. As usual, give it positive orientation, as shown in the picture.

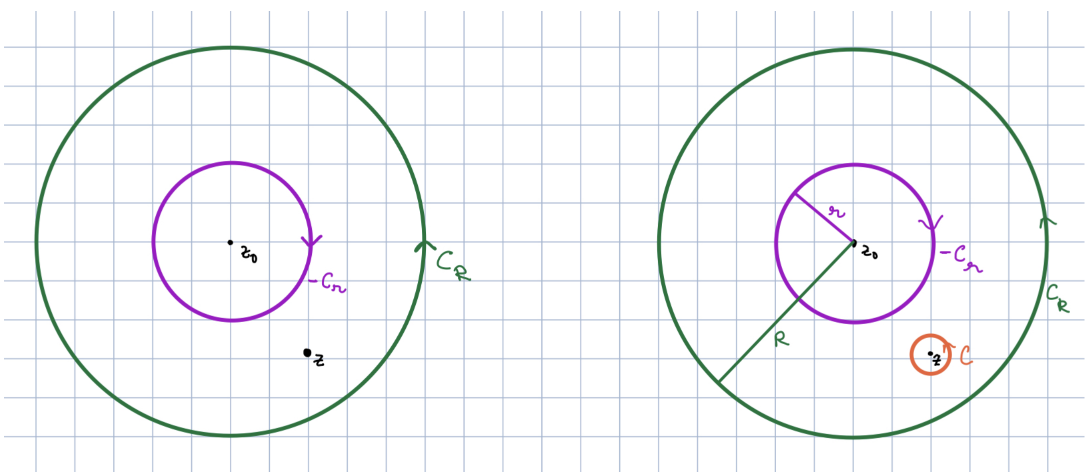

FIGURE 2. The direction shown on the smaller circle is negative, which is why the purple directed circles are labelled $-C_r$.

Next consider FIGURE 3. Here we introduce three line segments, one joining C_R and C_r (the line segment in red which is vertical), one path joining C_R and C (again in red) and one joining C_r and C (yet again in red).

Let Γ_L be the simple loop bounding coloured region on the picture of the annulus on the left in FIGURE 4, and Γ_R the corresponding loop bounding the coloured region on the picture of the annulus on the right. The orientations we give are the positive orientations, as shown in the pictures.

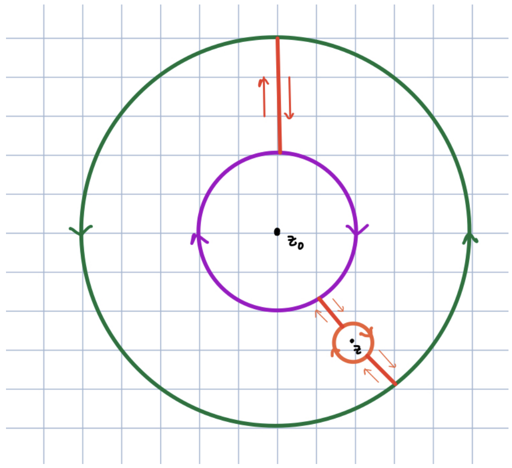

FIGURE 3. Dividing A into three simply connected subdivisions.

It is clear that for any function g, continous on Γ_L and Γ_R we have

$$\int_{C_R} g(\zeta)d\zeta - \int_{C_r} g(\zeta)d\zeta - \int_{C} g(\zeta)d\zeta = \int_{\Gamma_L} g(\zeta)d\zeta + \int_{\Gamma_R} g(\zeta)d\zeta.$$

Since $f(\zeta)/(\zeta - z)$ is analytic in in the interior of Γ_L and Γ_R we have the relation $\int_{\Gamma_L} f(\zeta)/(\zeta - z)d\zeta + \int_{\Gamma_R} f(\zeta)/(\zeta - z)d\zeta = 0$, whence

$$\int_{C_R} \frac{f(\zeta)}{\zeta - z}d\zeta - \int_{C_r} \frac{f(\zeta)}{\zeta - z}d\zeta - \int_{C} \frac{f(\zeta)}{\zeta - z}d\zeta = 0$$

Finally, we know that $(2\pi i)^{-1} \int_C f(\zeta)/(\zeta - z)d\zeta = f(z)$ (from the Cauchy Integral Formula). This proves (2.2.1).

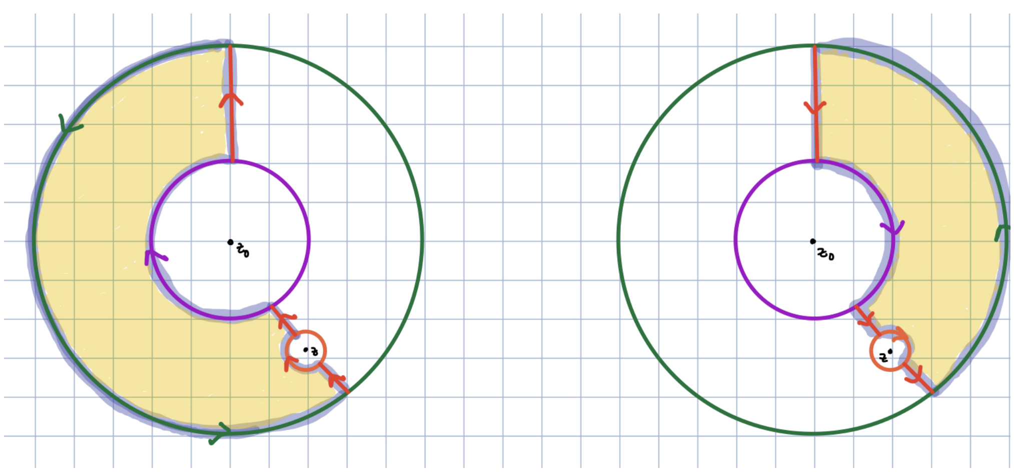

FIGURE 4. On the picture on the left, the yellow region is bounded by Γ_L, and on the picture on the right, the yellow region is bounded by Γ_R. The orientations of the simple loops Γ_L and Γ_R are positive.

Appendix A. **Radius of Convergence**

We will prove the following result

Theorem A.1. *Let* $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *be a power series, and suppose* $\lim_{n\to\infty} |a_n|^{1/n}$ *exists. Then the radius of convergence R of the given power series is given by the formula*

$$R = \frac{1}{\lim_{n\to\infty} |a_n|^{1/n}}$$

with the understanding that if $\lim_{n\to\infty} |a_n|^{1/n} = 0$, *then* $R = \infty$.

Proof. It is sufficient to prove the statement for $z_0 = 0$ and we will assume this.

Let $L = \lim_{n\to\infty} |a_n|^{1/n}$. Let us first consider the case where $L \neq 0$. Suppose $|z| < 1/L$. Then $|z|L < 1$. Pick $\epsilon > 0$ so that $|z|L + \epsilon < 1$. Since $|z|L = \lim_{n\to\infty}(|z||a_n|^{1/n})$, we have a non-negative integer N such that $||z||a_n|^{1/n} - |z|L| < \epsilon$ for all $n \geq N$. This means that $|z||a_n|^{1/n} < |z|L + \epsilon < 1$ for all $n \geq N$. Thus for $n \geq N$

$$|a_n z^n| \leq (|z|L + \epsilon)^n.$$

Since $0 < |z|L + \epsilon < 1$, it follows that $\sum_{n=N}^{\infty}(|z|L+\epsilon)^n$ is convergent. By Theorem A.5 of Lecture 18, it follows that $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is convergent.

If $L = 0$, then $\lim_{n\to\infty}|z||a_n|^{1/n} = 0$ for every $z \in \mathbb{C}$. This means there exists $N \geq 0$ such that $|z||a_n|^{1/n} < 1 - \epsilon$ for all $n \geq N$. Arguing as above, we see that in this case $\sum_{n=N}^{\infty}|a_n z^n|$ is convergent, since $|a_n z^n| < (1-\epsilon)^n$.

Next suppose $z \in \mathbb{C}$ is such that $|z| > 1/L$ (if $L = 0$, there is no such z). We have to show that $\sum_{n\geq 0} a_n z^n$ diverges. We will argue by contradiction. Suppose the series converges. Then in Lecture 18, we showed that for all w such that $|w| < |z|$, the series $\sum_{n\geq 0}|a_n w^n|$ is convergent. Pick w such that $L^{-1} < |w| < |z|$. It follows that $|w|L > 1$. Let $\epsilon > 0$ be such that $|w|L > 1 + \epsilon$. Then there exists $N \geq 0$ such that $|w||a_n|^{1/n} > 1 + \epsilon$ for all $n \geq 1$. Then $\sum_{n\geq N}|a_n w^n| \geq \sum_{n\geq N}(1+\epsilon)^n = \infty$. Thus $\sum_{n\geq 0} a_n w^n$ is not absolutely convergent giving a contradiction. $\square$