LECTURE 18

Date of Lecture: March 22, 2022

Throughout this lecture, $B_r(z_0)$ will denote the open circular neighbourhood (a.k.a. open disc) of radius r centred at z_0, i.e.

$$B_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < r\}. \tag{1}$$

The circle of radius r centred at r_0 is denoted $C_r(z_0)$.

$$C_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| = r\}. \tag{2}$$

Finally the closed disc of radius r centred ar z_0 is

$$\overline{B}_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| \leq r\} = B_r(z_0) \cup C_r(z_0). \tag{3}$$

1. **Power Series**

1.1. **Radius of Convergence.** Let $z_0 \in \mathbb{C}$. An infinite series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{1.1.1}$$

is called a power series at z_0. Suppose the series converges at $z = z_1 \neq Z_0$. Let $s = |z_1 - z_0|$. Note that since $z_1 \neq z_0$, $s > 0$. Since the series $\sum_{n=0}^{\infty} a_n (z_1 - z_0)^n$ converges, therefore the sequence $\{a_n (z_1 - z_0)^n\}$ is bounded, i.e. there exists a real number D such that $|a_n (z_1 - z_0)^n| \leq M$ for all $n \geq 0$ (see Theorem A.2 in the appendix).

Let r be a positive number strictly less than s. Let $\varrho = r/s$. Note that $0 \leq \varrho < 1$.

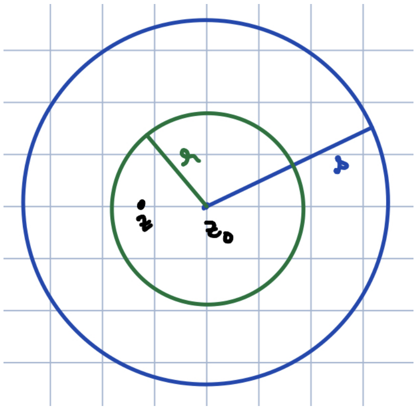

Figure 1.

Now suppose z is point in the disc $B_r(z_0)$ (see Figure 1). Then

$$|a_n (z - z_0)^n| = |a_n (z_1 - z_0)^n| |(z - z_0)^n / (z_1 - z_0)^n| \leq D \varrho^n. \tag{1.1.2}$$

Since $0 \leq \varrho < 1$, it follows that $\sum_{n=0}^{\infty} \varrho^n = 1/(1-\varrho) < \infty$, i.e. $\sum_{n=0}^{\infty} \varrho^n$ is convergent. By Theorem A.5 it follows that $\sum_{n=0}^{\infty} |a_n(z-z_0)^n|$ is convergent, i.e. $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is absolutely convergent.

Let $S = \{s \in [0,\infty) \mid \text{for some } z \in C_s(z_0), \sum_{n=0}^{\infty} a_n(z-z_0)^n\}$. Here, $C_s(z_0)$ is as in (2). In other words $C_s(z_0)$ is the circle of radius s centred at z_0. From what we have have proved, if $s \in S$, then $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges for all z in the disc $B_s(z_0)$. In other words, if $s \in S$, then $r \in$ for all $0 \leq r \leq s$. This means S is an interval, and hence is of the form $[0,b)$, $[0,b]$ with $b \in \mathbb{R}$, or $[0,\infty)$. We define a number R as follows. In the first two cases, set $R = b$, and in the last case, set $R = \infty$.

R as defined above is called *the radius of convergence of* $\sum_{n=0}^{\infty} a_n(z-z_0)^n$. Note that R could be 0 or ∞. It has the property that if $|z-z_0| < R$, then $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges and if $|z-z_0| > R$, then $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ diverges (i.e. it does not converge). We therefore have the following result

Proposition 1.1.3. *Let* $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *be a power series centred at* z_0*. There exists* $R \in [0,\infty]$ *(with* 0 *and* ∞ *included), called the* radius of convergence of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$*, such that if* $|z-z_0| < R$ *then* $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *converges, and if* $|z-z_0| > R$*, then* $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *diverges.*

Remark 1.1.4. The open disc $B_R(z_0)$ centred at z_0 is called the *disc of convergence*. We cannot say anything about the behaviour of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ on the circle $C_R(z_0)$.

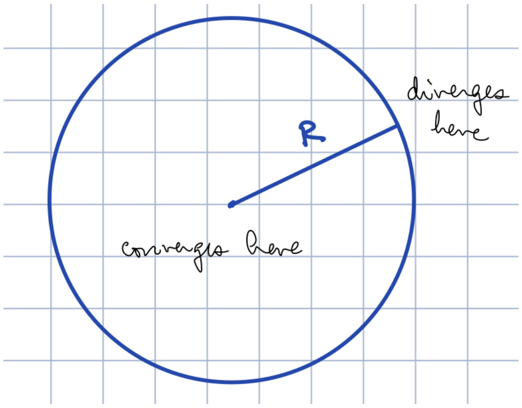

Figure 2. The power series $\sum_{n\geq 0} a_n(z-z_0)^n$ converges inside the disc $B_R(z_0)$, diverges outside. On the circle $C_R(z_0)$, it might converge at some points and diverge at others.

Now assume $R > 0$. Let r be such that $0 \leq r < R$. Pick s such that $r < s < R$. Note that since $0 \leq r < s < R$, both r and s are in S. As before, set $\varrho = r/s$, so that $0 \leq \varrho < 1$. The inequality (1.1.2) gives us another very important inequality, namely:

$$\begin{aligned}\left|\sum_{n=k+1}^{\infty} a_n(z-z_0)^n\right| &\le \sum_{n=k+1}^{\infty} |a_n(z-z_0)^n| \\ &\le \sum_{n=k+1}^{\infty} D\varrho^n \\ &= D\varrho^{k+1}\sum_{n=0}^{\infty}\varrho^n \\ &= D\frac{\rho^{k+1}}{1-\rho}.\end{aligned} \tag{1.1.5}$$

for all $z \in B_r(z_0)$. The crucial point is that (1.1.5) is true for every z in the closed disc $\overline{B}_r(z_0)$. Now clearly $\lim_{k\to\infty} D\varrho^{k+1}/(1-\varrho) = 0$, since $0 \le \varrho < 1$. This means given $\epsilon > 0$, there exists a non-negative integer K such that $D\frac{\rho^{k+1}}{1-\rho} < \epsilon$ for all $k \ge K$. By (1.1.5) we get

$$\left|\sum_{n=k+1}^{\infty} a_n(z-z_0)^n\right| < \epsilon, \qquad k \ge K, \quad z \in \overline{B}_r(z_0). \tag{1.1.6}$$

The important point is that K depends only on ϵ and not on $z \in \overline{B}_r(z_0)$.

Theorem 1.1.7. *Let* R *be the radius of convergence of* $\sum_{n=0}^{\infty} a_n(z-z_0)^n$*, and suppose* $R > 0$*. Define* $f\colon B_R(z_0) \to \mathbb{C}$ *by the formula* $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$*. Then*

(a) f *is analytic on the disc of convergence* $B_R(z_0)$*.*

(b) *For* z *in the disc of convergence* $B_R(z_0)$*, the derivative of* f *is given by term by term differentiation, i.e.*

$$f'(z) = \sum_{n=1}^{\infty} na_n(z-z_0)^{n-1} = \sum_{n=0}^{\infty} a_{n+1}(z-z_0)^n \qquad (z \in B_R(z_0).$$

(c) $f^{(n)}(z_0) = n!a_n$*.*

Proof. For (a) the strategy is to first show that f is continuous on $B_R(z_0)$ and then use Morera's theorem to prove analyticity. Let us carry out this strategy. Let $w \in B_R(z_0)$. Let us show that f is continuous at $z = w$. First, we know that $|w| < R$. Let r be a number such that $|w| < r < R$. Let $\epsilon > 0$ be given. We know, from (1.1.6) that there exists a non-negative integer K such that

$$\left|\sum_{n=k+1}^{\infty} a_n(z-z_0)^n\right| < \epsilon, \tag{$*$}$$

for $k \ge K$ and every $z \in \overline{B}_r(z_0)$. Let $P(z)$ be the polynomial

$$P(z) = \sum_{n=0}^{K} a_n(z-z_0)^n.$$

Since P is a polynomial, it is continuous. This means there exits $\delta > 0$ such that

$$|P(z) - P(w)| < \epsilon, \tag{$**$}$$

for every z such that $|z-w|<\delta$. This gives, using $(*)$ and $(**)$,

$$\begin{aligned}|f(z)-f(w)| &= |(f(z)-P(z))+(P(z)-P(w))+(P(w)-f(w)| \\ &\leq |f(z)-P(z)|+|P(z)-P(w)|+|f(w)-P(w)| \\ &= \left|\sum_{n=K+1}^{\infty} a_n(z-z_0)^n\right| + |P(z)-P(w)| + \left|\sum_{n=K+1}^{\infty} a_n(w-z_0)^n\right| \\ &< \epsilon+\epsilon+\epsilon = 3\epsilon\end{aligned}$$

for every z such that $|z-w|<\delta$. Thus f is continuous.

We now prove analyticity. Let Γ be an closed loop in the disc of convergence $B_R(z_0)$. Let r be a number such that $0<r<R$ and Γ lies in the disc $B_r(z_0)$. Let $\epsilon>0$ be given. Then we have a non-negative integer K such that the inequality (1.1.6) is true for all $k\geq K$ and all $z\in B_r(z_0)$. As before let $P(z)=\sum_{n=0}^{K} a_n(z-z_0)^n$. Since P is a polynomial, therefore it is analytic, and hence by Cauchy's Integral Theorem $\int_\Gamma P(z)dz=0$. We thus have

$$\left|\int_\Gamma f(z)dz\right| = \left|\int_\Gamma f(z)dz-\int_\Gamma P(z)dx\right| \leq \int_\Gamma\left|\sum_{n=K+1}^{\infty} a_n(z-z_0)^n\right|dz \leq \epsilon\ell(\Gamma).$$

Thus, for every $\epsilon>0$, $|\int_\Gamma f(z)dz|\leq\epsilon\ell(\Gamma)$. This means $\int_\Gamma f(z)dz=0$. By Morera's Theorem (see Theorem 1.1.6 of Lecture 16), f is analytic.

Let us now prove (b). It is enough to prove the formula for $z\in B_c(z_0)$ for every $0\leq c<R$. Fix c as above and let $z\in B_c(z_0)$. Choose r such that $0\leq c<r<R$, Let $\epsilon>0$ be given. We can find a non-negative integer such that (1.1.6) holds for all $k\geq K$ and all points in $\overline{B}_r(z_0)$. For $k\geq K$, let P_k be the polynomial

$$P_k(z)=\sum_{n=0}^{k} a_n(z-z_0)^n.$$

Observe that if $\zeta\in C_r(z_0)$ and $z\in B_c(z_0)$, then $|\zeta-z|>r-c$ and hence

$$(\#) \qquad \frac{1}{|\zeta-z|^2}<\frac{1}{(r-c)^2}.$$

For $k\geq K$ and $z\in B_c(z_0)$ we have

$$\begin{aligned}|f'(z)-\sum_{n=1}^{k} na_nz^{n-1}| &= |f'(z)-P_k'(z)| \\ &= \left|\frac{1}{2\pi i}\oint_{C_r(z_0)}\frac{f(\zeta)}{(\zeta-z)^2}d\zeta-\frac{1}{2\pi i}\oint_{C_r(z_0)}\frac{P_k(\zeta)}{(\zeta-z)^2}d\zeta\right| \\ &\leq \left|\frac{1}{2\pi i}\oint_{C_r(z_0)}\frac{f(\zeta)-P_k(z)}{(\zeta-z)^2}d\zeta\right| \\ &\leq \frac{1}{2\pi}\oint_{C_r(z_0)}\left|\frac{\sum_{n=k+1}^{\infty}a_n(z-z_0)^n}{(\zeta-z)^2}\right|d\zeta \\ &< \frac{\epsilon}{2\pi(r-c)^2}(2\pi r)=\frac{\epsilon r}{(r-c)^2} \qquad \text{(by (1.1.6)) and } (\#)\end{aligned}$$

By definition of limits this means $\lim_{k\to\infty}\sum_{n=1}^{k} na_nz^{n-1}=f'(z)$. Thus $f'(z)=\sum_{n=1}^{\infty} na_nz^{n-1}$, which is what we were asked to prove.

Part (c) follows from (b). $\square$

1.2. **Taylor and Maclaurin series.** Let f be analytic on a domain D. Let $z_0 \in D$. The power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}}{n!}(z_0)(z-z_0)^n \tag{1.2.1}$$

is said to be the *Talyor series* of f at z_0. If $z_0 = 0$, then the series is called the *Maclaurin series* of f.

We will now show that the Taylor's series for f at z_0 converges in every open disc centred at z_0 and contained in D. Let $B_s(z_0) \subset D$. To show convergence in $B_s(z_0)$, it is enough to show show convergence in $B_r(z_0)$ for all $0 \le r < s$. Now if $0 \le r < s$, then the closed disc $\overline{B}_r(z_0)$ is contained in D. Thus it is enough to prove convergece of the Taylor's series (1.2.1) for closed discs contained in D and centred at z_0. Therefore, without loss of generality, assume $\overline{B}_s(z_0) \subset D$, and let us prove the convergence of (1.2.1) on $B_s(z_0)$.

Without loss of generality, let $z_0 = 0$. Let $z \in B_s(0)$. Pick r such that $|z| < r < s$. Let $\varrho = r/s < 1$. Then for every $\zeta \in C_r(0)$, we have $|z/\zeta| < \varrho < 1$. Since $0 \le \varrho < 1$, we have that the series $\sum_{n=0}^{\infty} \varrho^n$ converges and in fact $\sum_{n=0}^{\infty} \varrho^n = 1/(1-\varrho)$. By part (c) of Theorem A.4, this means that $\lim_{k\to\infty} \sum_{n=k+1}^{\infty} \varrho^n = 0$.

Now suppose $\epsilon > 0$ is given. By definition of a limit, the above shows that there exists a non-negative integer K such that

$$\left| \sum_{n=k+1}^{\infty} \varrho^n \right| < \epsilon, \quad \text{for all } k \ge K. \tag{1.2.2}$$

Next let $\zeta \in C_r(0)$. Then, as $|z/\zeta| < \varrho < 1$, we have

$$\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta)}{\zeta} \frac{1}{(1-(z/\zeta))} = \frac{f(\zeta)}{\zeta} \sum_{n=0}^{\infty} (z/\zeta)^n = \sum_{n=0}^{\infty} \frac{f(\zeta)}{\zeta^{n+1}} z^n.$$

Let M be the maximum value of $|f(\zeta)|$ for ζ on the circle $C_r(0)$. From (1.2.2), we see that for $k \ge K$, and $\zeta \in C_r(0)$

$$\left| \frac{f(\zeta)}{\zeta - z} - \sum_{n=0}^{k} \frac{f(\zeta)}{\zeta^{n+1}} z^n \right| = \left| \frac{f(\zeta)}{\zeta} \sum_{n=k+1}^{\infty} (z/\zeta)^n \right| \le \frac{M}{r} \sum_{n=k+1}^{\infty} \varrho^n < M\epsilon/r. \tag{1.2.3}$$

Thus

$$\begin{aligned} |f(z) - \sum_{n=0}^{k} (f^{(n)}(0)/n!) z^n| &= \left| \frac{1}{2\pi i} \oint_{C_r(0)} \left(\frac{f(\zeta)}{\zeta - z} - \sum_{n=0}^{k} \frac{f(\zeta)}{\zeta^{n+1}} z^n \right) d\zeta \right| \\ &< M\epsilon(2\pi r)/(2\pi r) \\ &= M\epsilon \end{aligned}$$

for $k \ge K$. Thus

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

for all $z \in B_r(0)$.

There was nothing special about $z_0 = 0$. The above computations give:

Theorem 1.2.4. *Let f be analytic on a domain D. Let $z_0 \in D$. Let $B_r(z_0)$ be an open disc centred at z_0 such that $B_r(z_0) \subset D$. Then the Taylor series $\sum_{n=0}^{\infty}(f^{(n)}(z_0)/n!)(z-z_0)^n$ converges in $B_r(z_0)$ and*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

for $z \in B_r(z_0)$.

Appendix A. **Basic results on convergence**

In this appendix we gather together the results we need on convergence of sequences and series. This is for ready reference. You are expected to have seen these results in earlier courses, at least when the sequences and series in question were over the real numbers. The same proofs work over complex numbers. You won't be tested on these, but you might be expected to use the results to prove results about power series, contour integrals, etc.

Definitions A.1. Let $\{x_n\}$ be a sequence of complex numbers and $\sum_{n=0}^{\infty} c_n$ an infinite series of complex numbers.

1. The sequence $\{x_n\}$ is said to be *bounded* if there exists a real number $M < \infty$ such that $|x_n| \le M$ for all n.
2. It is said to be *convergent* if it has a limit. Recall that this means there is complex number L such that for every $\epsilon > 0$, there exists $N \ge 0$ such that $|x_n - L| < \epsilon$ for every $n \ge N$. In this case we say L is the *limit* of $\{x_n\}$ as n approaches infinity, and write $\lim_{n\to\infty} x_n = L$.
3. It is said to be a *Cauchy sequence*, if given $\epsilon > 0$, there exists $N \ge 0$ such that $|x_n - x_m| < \epsilon$ for all $n \ge N$.
4. The series $\sum_{n=0}^{\infty} c_n$ is said to *converge* if the sequence $\{s_n\}$ defined by $s_n = \sum_{k=0}^{n} c_k = c_0 + \cdots + c_n$ converges. The number s_n is called the n^{th} partial sum of the series $\sum_{n=0}^{\infty} c_n$. If $c = \lim_{n\to\infty} s_n$, we say that the infinite series $\sum_{n=0}^{\infty} c_n$ *converges to* c and in this case we write $\sum_{n=0}^{\infty} c_n = c$. The number c is called the *sum* of the series $\sum_{n=0}^{\infty} c_n$, or more understandably, the sum of the c_n as n varies over the non-negative integers.
5. The series $\sum_{n=0}^{\infty} c_n$ is said to be *absolutely convergent* if the series $\sum_{n=0}^{\infty}|c_n|$ is convergent. In Theorem A.5, it is shown that an absolutely convergent series is convergent. The other way around may not be true.

Theorem A.2. *If $\{x_n\}$ is a convergent sequence, then it is bounded.*

Proof. Let $L = \lim_{n\to\infty} x_n$. Take $\epsilon = 1$. There exists $N \ge 0$ such that $|x_n - L| < \epsilon$ for $n \ge N$. Since $|x_n| - |L| \le |x_n - L|$, we see that $|x_n| - |L| < \epsilon$ for $n \ge N$, which in turn means that $|x_n| < |L| + \epsilon$ for $n \ge N$. On the other hand, the finite set $\{x_0, x_1, \ldots, x_{N_1}\}$ is clearly bounded, for example by $m = \max\{|x_0|, |x_1|, \ldots, |x_{N_1}|\}$. Let $M = \max\{m, |L| + \epsilon\}$. It is clear that $|x_n| \le M$ for all $n \ge 0$. Thus the given sequence is bounded. ☐

Theorem A.3. *A sequence is convergent if and only if it is Cauchy.*

Proof. Suppose $\{x_n\}$ is convergent, and let $\lim_{n\to\infty} x_n = L$ (say). Given $\epsilon > 0$, there exists $N \ge 0$ such that $|x_n - L| < \epsilon/2$. Now suppose $n, m \ge N$. Then $|x_n - x_m| = |(x_n - L) - (x_m - L)| \le |x_n - L| + |x_m - L| < \epsilon/2 + \epsilon/2 = \epsilon$. Thus $\{x_n\}$ is Cauchy.

The proof of the converse is omitted. Itrequires a fundamental property of the real numbers (the so called "least upper bound property"). In some approaches, the convergence of Cauchy sequences is built into the definition of real numbers (this is Cantor's approach to the construction of the real numbers). ☐

Theorem A.4. *Let* $\sum_{n=0}^{\infty} c_n$ *be a convergent series, say* $\sum_{n=0}^{\infty} c_n = c$*.*

(a) *For* $k \geq 0$*, the series* $\sum_{n=k}^{\infty} c_n = c - \sum_{n=0}^{k-1} c_n$*, i.e. the series* $\sum_{n=k}^{\infty} c_n$ *converges and its sum is* $c - \sum_{n=0}^{k-1} c_n$*.*

(b) *The sequence* $\{c_n\}$ *converges to zero, i.e.* $\lim_{n\to\infty} c_n = 0$*.*

(c) $\lim_{k\to\infty} \sum_{n=k}^{\infty} c_n = 0$.

Proof. Part (a) is obvious.

For (b), note that if $\{s_n\}$ is the sequence of partial sums of $\sum_{n\geq 0} c_n$, then $c_n = s_n - s_{n-1}$. Then $\lim_{n\to\infty} c_n = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = c - c = 0$.

Part (c) is seen as follows. We have $\sum_{n=k}^{\infty} c_n = c - \sum_{n=0}^{k-1} c_n$. Letting $k \to \infty$, we see that $\sum_{n=k}^{\infty} c_n \to c - \lim_{k\to\infty} \sum_{n=0}^{k-1} c_n = c - c = 0$. ☐

Theorem A.5. *Let* $\sum_{n=0}^{\infty} c_n$ *and* $\sum_{n=0}^{\infty} r_n$ *be infinite series of complex numbers such that* r_n *is real for all* n*, and* $|c_n| \leq r_n$ *for every* $n \geq 0$*. Suppose* $\sum_{n=0}^{\infty} r_n$ *is convergent. Then* $\sum_{n=0}^{\infty} c_n$ *is convergent. In particular, an absolutely convergent series is convergent (take* $r_n = |c_n|$*).*

Proof. Let $\{s_n\}$ be the sequence of partial sums for $\sum_{n=0}^{\infty} c_n$ and $\{\sigma_n\}$ the sequence of partial sums for $\sum_{n=0}^{\infty} r_n$. In other words, let $s_n = c_0 + \cdots + c_n$ and $\sigma_n = r_0 + \ldots + r_n$. Since $\sum_{n=0}^{\infty} r_n$ is convergent, the sequence $\{\sigma_n\}$ is convergent. By Theorem A.3, it is Cauchy. thus, given $\epsilon > 0$, there exists $N \geq 0$ such that $|\sigma_n - \sigma_m| < \epsilon$ for $m, n \geq N$. Without loss of generality, we may let $m \leq n$. Then $s_n - s_m = \sum_{k=m}^{n} c_k$ and $\sigma_n - \sigma_m = \sum_{k=m}^{n} r_k$. Thus

$$|s_n - s_m| = |\sum_{k=m}^{n} c_k| \leq \sum_{k=m}^{n} |c_k| \leq \sum_{k=m}^{n} r_k = |\sigma_n - \sigma_m| < \epsilon$$

for $n, m \geq N$. This means $\{s_n\}$ is Cauchy, and hence by Theorem A.3, $\{s_n\}$ is convergent. By definition, this means that the infinite series $\sum_{n=0}^{\infty} c_n$ is convergent. ☐