LECTURE 17

Date of Lecture: March 17, 2022

1. Some examples of Contour Integration

1.1. Recall the following formula from the last lecture, where Γ is a simple loop, z a point in the interior of Γ, and f a function analytic at each point of Γ and in the interior of Γ.

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}d\zeta \tag{1.1.1}$$

1. Let Γ be a simple loop in $\mathbb{C}$. Evaluate $\int_\Gamma \dfrac{e^{2z}}{(z-w)^5}dz$ for $w \notin \Gamma$.

Solution: Let $f(z) = e^{2z}$. Suppose first that w in in the interior of Γ. By (1.1.1), $\int_\Gamma f(z)/(z-w)^5 dz = ((2\pi i)/(4!))f^{(4)}(w)$, and hence

$$\int_\Gamma \frac{e^{2z}}{(z-w)^5}dz = \frac{2\pi i}{24}\frac{d^4}{dz^4}e^{2z}\bigg|_{z=w} = \frac{2\pi i}{24}(2^4)e^{2w} = \frac{4\pi}{3}e^{2w}.$$

If w is in the exterior of Γ, by the Cauchy Integral Theorem,

$$\int_\Gamma \frac{e^{2z}}{(z-w)^5}dz = 0.$$

Thus

$$\int_\Gamma \frac{e^{2z}}{(z-w)^5}dz = \begin{cases} \dfrac{4\pi}{3}e^{2w} & \text{if } w \text{ is in the interior of } \Gamma \\ \\ 0 & \text{if } w \text{ is in the exterior of } \Gamma. \end{cases}$$

2. Let Γ be a simple loop in $\mathbb{C}$. Evaluate $\int_\Gamma \dfrac{z^7+3z^6+2z+1}{(z-w)^7}dz$ for $w \notin \Gamma$.

Solution: If w is in the interior of Γ, then following the techinique of the previous example we see that

$$\begin{aligned}\int_\Gamma \frac{z^7+3z^6+2z+1}{(z-w)^7}dz &= \frac{2\pi i}{6!}\frac{d^6}{dz^6}(z^7+3z^6+2z+1)\bigg|_{z=w} \\ &= \frac{2\pi i}{6!}((7!)w + 3(6!)) = (2\pi i)(7w+3).\end{aligned}$$

If w is in the exterior of Γ, the given integral is zero by the Cauchy Integral Theorem. So

$$\int_\Gamma \frac{z^7+3z^6+2z+1}{(z-w)^7}dz = \begin{cases} (2\pi i)(7w+3) & \text{if } w \text{ is in the interior of } \Gamma \\ \\ 0 & \text{if } w \text{ is in the exterior of } \Gamma. \end{cases}$$

2. **The maximum modulus property**

2.1. **The averaging property.** Let $z_0 \in \mathbb{C}$, R a positive number, and f function analytic in the disc B given by $|z - z_0|R$ and at every point of the circle C given by $|z - z_0| = R$. We orient C positively. By the Cauchy Integral formula, giving we have $f(z_0) = 1/(2\pi i)\int_C f(z)/(z-z_0)dz$. Using the parametrization $z(t) = z_0 + Re^{it}$, $t \in [0, 2\pi]$ for C, we see that $f(z_0) = (2\pi i)^{-1}\int_0^{2\pi} f(z_0 + Re^{it})/(Re^{it})(iRe^{it})dt = 1/(2\pi)\int_0^{2\pi} f(z_0 + Re^{it})dt$. In other words

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{it})dt. \tag{2.1.1}$$

In other words the value of f at the centre of the circle C is the average of the value of f on the circle. This is known as the *averaging property* of analytic functions.

Suppose $M = \max\{|f(z)| \mid z \in C\}$. Then $|f(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi}|f(z_0 + Re^{e^i t})|dt \leq \frac{1}{2\pi}M\int_0^{2\pi} dt = M$. Thus

$$|f(z_0)| \leq \max_{z\in C}|f(z)|. \tag{2.1.2}$$

Theorem 2.1.3. (The Maximum Modulus Principle) *Let f be an analytic function on a domain D. Suppose there is a point z_0 in D such that $|f|$ attains its maximum at z_0, i.e. $|f(z)| \leq |f(z_0)|$ for all $z \in D$. Then f is constant on D.*

Proof. From one of the homework problems (Problem 2.4.12 from the text) you did, it is enough to prove that $|f|$ is a constant. Let $M = |f(z_0)|$. We claim that $|f(z)| = M$ for all $z \in D$. Let us first prove it in a neighbourhood of z_0 in D.

Let R be a real number such that the closed circular neighbourhood of radius R around z_0 lies in D, i.e. R is chosen so that $\{z \in \mathbb{C} \mid |z - z_0| \leq R\}$ is contained in $\mathbb{C}$. Let C be the bounding circle, i.e. $C = \{z \in \mathbb{C} \mid |z - z_0| = R\}$. Suppose we have a point ζ on the circle C such that $|f(\zeta)| \neq M$. Since M is the maximum of $|f|$ on D, this means $|f(\zeta)| < M$.

Let $h(t) = |f(z_0 + Re^{it})|$, $t \in [0, 2\pi]$. Then h is continuous. Now $\zeta = z_0 + Re^{i\theta}$ for some $\theta \in [0, 2\pi]$. Moreover, $h(\theta) = |f(z_0 + Re^{i\theta})| = |f(\zeta)|$. Since $|f(\zeta)| < M$, we have $h(\theta) < M$. By the continuity of h there is an interval $I = [\theta - \delta, \theta + \delta]$[1] around θ on which $h(t) < M$. The average of h over $[0, 2\pi]$ must be less than M, since the contribution to the average from the interval I is less than M, and there is no way to compensate for this on the remaining part of the interval $[0, 2\pi]$, for $h(t)$ cannot exceed M. In other words, in this case

$$M = |f(z_0)| \leq 1/(2\pi)\int_0^{2\pi}|f(z + Re^{it})|dt = 1/(2\pi)\int_0^{2\pi} h(t)dt < M.$$

i.e. $M < M$, which is a contradiction. It follows that $|f(z)| = M$ for every point on z on C. The same reasoning shows that $|f(z)| = M$ on every circle centred at z_0 with radius less than or equal to R. It follows that $|f(z)| = M$ for every z in the circular neighbourhood of radius R centred at z_0. All we require of the number R is that the closed disc of radius R centred at z_0 lies entirely in D.

Let U be the set of points $w \in D$ such that $|f(w)| = M$. Then by the above reasoning, around each point of U there is a circular neighbourhood lying entirely in U, which means U is open.

[1] if $\theta = 0$, $I = [0, \delta]$ and if $\theta = 2\pi$, $I = [2\pi - \delta, 2\pi]$.

Let $V = \{z \in D \mid |f(z)| < M\}$. Since $|f|$ is continuous, it is easy to see that V is also open, for if z in D is such that $|f(z)| < M$, then there is a circular neighbourhood of z with the property that if w lies in this neighbourhood, then $|f(w)| < M$. Now clearly $U \cup V = D$ and $U \cap V = \emptyset$. From Theorem A.1 in the appendix below, since $U \neq \emptyset$, we must have $V = \emptyset$, i.e. $U = D$. This means that $|f(z)| = M$ for every $z \in D$. We have therefore proved that $|f|$ is a constant. As mentioned above, by Problem 2.4.12 of the text, this means f is a constant. □

Appendix A.

Here is a slightly advanced result which is very useful. We have already used it in the proof of the maximum modulus principle above.

Theorem A.1. *Let D be a domain in $\mathbb{C}$ and U, V open subsets of D such that $U \cup V = D$ and $U \cap V = \emptyset$. Then either U or V is empty.*

Proof. Define a function $\varphi\colon D \to \mathbb{R}$ by the rule

$$\varphi(z) = \begin{cases} 0 & \text{if } z \in U \\ 1 & \text{if } z \in V. \end{cases}$$

Then φ is continuous, for, given a point $z \in D$ we can find an open neighbourhood of z on which φ is contant and hence continuous. Suppose U and V are both non-empty. Pick $z_0 \in U$ and $z_1 \in V$. Since D is a domain, we can find a continuous path connecting z_0 and z_1, in other words, we can find a continuous function $\gamma\colon [a,b] \to D$ such that $\gamma(a) = z_0$ and $\gamma(b) = z_1$. Let $h = \varphi \circ \gamma$, i.e. let $h\colon [a,b] \to \mathbb{R}$ be the function given by the rule $h(t) = \varphi(\gamma(t))$ for $t \in [a,b]$. Then h is continuous, being the composite of two continuous functions. Now, $h(a) = 0$ and $h(b) = 1$. Clearly, for $t \in [a,b]$, $h(t)$ is either 0 or 1, and nothing in between. This contradicts the Intermediate Value Theorem. Thus one of U or V must be empty. □