LECTURE 16

Date of Lecture: March 15, 2022

1. Infinite differentiability of analytic functions

1.1. **A general formula for derivatives.** Here is a more general setting than what we had in the last lecture. Let Γ be a contour (not necessarily closed) and

$$\phi \colon \Gamma \longrightarrow \mathbb{C}$$

a continuous function on Γ. On $D = \mathbb{C} \smallsetminus \Gamma$ define the functions F_n, one for each non-negative integer n, as follows

$$F_n(z) := \int_\Gamma \frac{\phi(\zeta)}{(\zeta - z)^{n+1}} d\zeta, \qquad z \notin \Gamma. \tag{1.1.1}$$

Theorem 1.1.2. *F_n is analytic on $\mathbb{C} \smallsetminus \Gamma$ for every $n \geq 0$, and for such n, the following formula holds*

$$F_n'(z) = (n+1)F_{n+1}(z) \qquad z \in \mathbb{C} \smallsetminus \Gamma.$$

In particular F_0 is infinitely differentiable on $\mathbb{C} \smallsetminus \Gamma$ and for $n \geq 0$

$$F_0^{(n)}(z) = n! F_n(z) \qquad z \in \mathbb{C} \smallsetminus \Gamma.$$

Proof. The second formula follows from the first in an obvious. We will now prove the first formula (at least for $n = 0$).

Pick $z \in \mathbb{C} \smallsetminus \Gamma$. There exists a circular neighbourhood z,

$$B = \{w \in \mathbb{C} \mid |w - z| < r\},$$

such that B lies in $\mathbb{C} \smallsetminus \Gamma$, and further the circle $C_r = \{w \mid |w - z| = r\}$ also lies in $\mathbb{C} \smallsetminus \Gamma$. Since Γ is closed and bounded and the closed disc $\overline{B} = B \cup C_r$ is also closed and bounded, we have positive real numbers δ and Δ defined by $\delta = \min\{|\zeta - w| \mid \zeta \in \Gamma, w \in \overline{B}\}$ and $\Delta = \max\{|\zeta - w| \mid \zeta \in \Gamma, w \in \overline{B}\}$. Then $0 < \delta < \Delta < \infty$. Let

$$M = \max_{\zeta \in \Gamma} |\phi(\zeta|. \tag{$*$}$$

We will show that F_0 is differentialble in $\mathbb{C} \smallsetminus \Gamma$ and that $F_0' = F_1$ on this set. To that end, here are some computations.

$$\begin{aligned}
\frac{1}{h}\left\{\frac{1}{\zeta - z - h} - \frac{1}{\zeta - z}\right\} - \frac{1}{(\zeta - z)^2} &= \frac{1}{h}\frac{(\zeta - z) - (\zeta - z - h)}{(\zeta - z)(\zeta - z - h)} - \frac{1}{(\zeta - z)^2} \\
&= \frac{1}{(\zeta - z)(\zeta - z - h)} - \frac{1}{(\zeta - z)^2} \\
&= \frac{(\zeta - z) - (\zeta - z - h)}{(\zeta - z)^2(\zeta - z - h)} \\
&= \frac{h}{(\zeta - z)^2(\zeta - z - h)}.
\end{aligned}$$

Pick h so that $|h| < r$. This means $z+h$ lies in B. For such h, by definition of δ, $|\zeta - z - h| > \delta$. We also have, since $z \in B$, $|\zeta - z| > \delta$. This means that $|(\zeta - z)^2(\zeta - z - h| > \delta^3$ and hence

$$(**) \qquad \left| \frac{1}{h}\left\{ \frac{1}{\zeta - z - h} - \frac{1}{\zeta - z} \right\} - \frac{1}{(\zeta - z)^2} \right| \leq \frac{|h|}{\delta^3}.$$

Using $(*)$ and $(**)$ we get

$$(\dagger) \qquad \left| \frac{1}{h}\left\{ \frac{\phi(\zeta)}{\zeta - z - h} - \frac{\phi(\zeta)}{\zeta - z} \right\} - \frac{\phi(\zeta)}{(\zeta - z)^2} \right| \leq \frac{|h|M}{\delta^3}.$$

This means

$$\left| \int_\Gamma \left[\frac{1}{h}\left\{ \frac{\phi(\zeta)}{\zeta - z - h} - \frac{\phi(\zeta)}{\zeta - z} \right\} - \frac{\phi(\zeta)}{(\zeta - z)^2} \right] d\zeta \right| \leq \frac{|h|M}{\delta^3}\ell(\Gamma).$$

Unravelling the above inequality, we get, for h such that] $|h| < r$, the following inequality.

$$\left| \frac{F_0(z+h) - F_0(z)}{h} - F_1(z) \right| \leq \frac{|h|M}{\delta^3}\ell(\Gamma).$$

Letting $h \to 0$, we see that F_0' exists on $\mathbb{C} \smallsetminus \Gamma$, and on this set, $F_0'(z) = F_1(z)$.

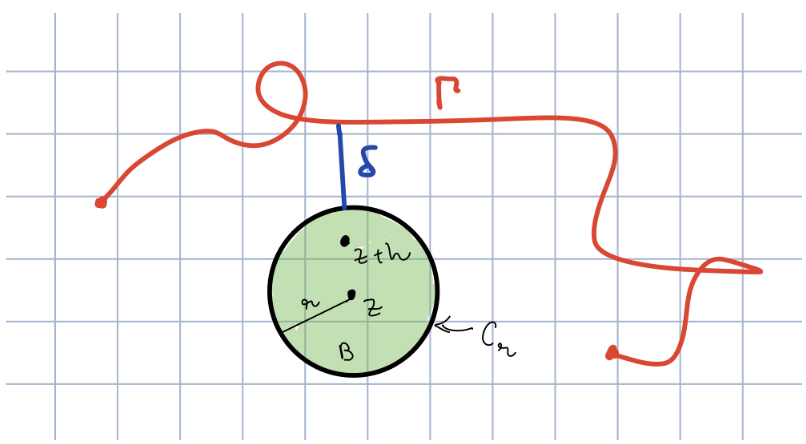

FIGURE 1. Γ need not be a loop. The point z is the centre of the circle C_r of radius r, and h is chosen so that $|h| < r$ so that $z+h$ lies inside the circle, i.e. in the disc B. The shortest distance between Γ and C_r is δ.

The same technique works for $n > 1$ too. The details are left to you. For example, one can show that

$$(\ddagger) \qquad \frac{1}{h}\left\{ \frac{1}{(\zeta - z - h)^2} - \frac{1}{(\zeta - z)^2} \right\} - \frac{2}{(\zeta - z)^3} = \frac{3(\zeta - z)h - 2h^2}{(\zeta - z - h)^2(\zeta - z)^3}$$

By the triangle inequality, one gets

$$\left| \frac{3(\zeta - z)h - 2h^2}{(\zeta - z - h)^2(\zeta - z)^3} \right| \leq \frac{(3\Delta + 2|h|)|h|}{\delta^5}.$$

From here one sees (by multiplying the expressions on both sides of (‡) by $\phi(\zeta)$ and then intgrating over Γ) that

$$\left|\frac{F_1(z+h)-F_1(z)}{h}-2F_2(z)\right| \leq \frac{(3\Delta+2|h|)|h|}{\delta^5}M\ell(\Gamma)$$

from which it is easy to see (by letting $h \to 0$) that F_1' exists and equals $2F_2$ on $\mathbb{C} \smallsetminus \Gamma$. The general case is similar.

For a very elegant proof without messy computations (but requiring some mathematical maturity to read) see [A, p.121, Lemma 3] ☐

From Theorem 1.1.2 we deduce the following result.

Theorem 1.1.3. *Let Γ be. simple loop and f a function which is analytic in the interior of Γ and on all point of Γ. Then f is infinitely differentiable in the interior of Γ and*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(\zeta)}{(\zeta-z)^{n+1}}d\zeta,$$

for all z in the interior of Γ.

Proof. In the statement of Theorem 1.1.2, take ϕ to be the function $1/(2\pi i)f$, restricted to Γ. By Cauchy's integral formula $F_0 = f$ in the interior Γ. The rest follows from Theorem 1.1.2 ☐

Theorem 1.1.4. *Let f be analytic on a domain D. Then f is infinitely differentiable on D.*

Proof. Let z be a point in D. Pick a circular neighbourhood B of z such that B and the bounding circle C of B lie in D. Then in B, from Theorem 1.1.3, f is infinitely differentiable. Since the property of being infinitely differentiable is local, f is infinitely differentiable. ☐

The following corollary is immediate.

Corollary 1.1.5. *Let D be a domain, f a function on D which has an antiderivative. Then f is analytic on D.*

Proof. Let F be an antiderivative of f on D. Then F is infinitely differentiable according to Theorem 1.1.4. In particular F' is analytic on D. Since $f = F'$ we are done. ☐

Corollary 1.1.5 gives us a well known theorem known as *Morera's Theorem.*

Theorem 1.1.6. (Morera's Theorem)*Let f be a continuous function on a domain D such that $\int_\Gamma f(z)dz = 0$ for every loop Γ in D. Then f is analytic.*

Proof. The condition on f is equivalent to saying f has an antiderivative in D (see Theorem 2.1.2 in Lecture 13). By Corollary 1.1.5, f is analytic on D. ☐

2. Liouville's Theorem and the Fundamental Theorem of Algebra

2.1. **The Cauchy estimates.** Let $z \in \mathbb{C}$ and let $B_R = B_R(z)$ be the circular neighbourhood $B_R = \{w \in \mathbb{C} \mid |w-z| < R\}$, and $C_R = \{\zeta \in \mathbb{C} \mid |\zeta - z| = R\}$ the bounding circle of B_R. Suppose f is analytic on B_R and also at each point of C_R. From Theorem 1.1.3, we get

$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_{C_R} \frac{f(\zeta)}{(\zeta-z)^{n+1}}d\zeta.$$

Let M be the maximum of $|f(\zeta)|$ as ζ varies over C_R. Then, for $\zeta \in C_R$, $|f(\zeta)/(\zeta - z)^{n+1}| = |f(\zeta)/R^{n+1}| \leq M/R^{n+1}$. It follows that

$$|f^{(n)}(z)| \leq \frac{n!}{2\pi} \frac{M}{R^{n+1}} \ell(C_R) = \frac{n!}{2\pi} \frac{M}{R^{n+1}} 2\pi R = \frac{n!M}{R^n}.$$

The inequalities (one for each $n \geq 0$)

$$|f^{(n)}(z)| \leq \frac{n!M}{R^n} \tag{2.1.1}$$

are called the *Cauchy estimates.*

2.2. **Liouville's Theorem.** The Cauchy estimates are very useful. They give us Lousiville's Theorem which says that a non constant entire function cannot be bounded. Recall that a function f is said to be bounded in its domain D if there exists a number M such that $|f(z)| \leq M$ for all $z \in D$. We state Lousville's Theorem in the following form:

Theorem 2.2.1. (Louisville's Theorem) *Suppose f is an entire bounded function. Then f is a constant.*

Proof. For $z \in \mathbb{C}$, let C_R be the circle of radius R centred at z, oriented in the positive direction. Since f is bounded, there exists a number M such that $|f(z)| \leq M$ for $z \in \mathbb{C}$. Therefore by the Cauchy estimates

$$|f'(z)| \leq \frac{M}{R}.$$

Let $R \to \infty$. We see that $|f'(z)| = 0$, i.e. $f'(z) = 0$. Since $z \in \mathbb{C}$ was chosen arbitrarily, f' vanishes at every point of $\mathbb{C}$. Thus f is a constant. □

2.3. **The Fundamental Theorem of Algebra.** Let

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n \tag{2.3.1}$$

be a polynomial of degree n. This means $a_n \neq 0$. Suppose further that p is not a constant polynomial, i.e. suppose $n \geq 1$. Then a famous theorem of Gauss says that the equaltion $p(z) = 0$ has at least one solution. This is the so called *Fundamental Theorem of Algebra.*

Theorem 2.3.2. (The Fundamental Theorem of Algebra) *Let p be a polynomial with complex coefficients with degree of p at least 1. Then there exists a complex number w such that $p(w) = 0$, i.e. the equation $p(z) = 0$ has at least one solution.*

Proof. Write p as in (2.3.1). We know that $a_n \neq 0$. Now, on $\mathbb{C} \setminus \{0\}$,

$$p(z) = z^n \left(\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + a_n \right).$$

As $z \to \infty$, the expression in parentheses approaches $a_n \neq 0$. Since $z^n \to \infty$ as $z \to \infty$, the limit of $p(z)$ as z approaches ∞ is of the form $\infty \cdot a_n$, in other words $\lim_{z\to\infty} p(z) = \infty$. This means, there exists $R > 0$ such that $|p(z)| \geq 1$ for all z such that $|z| > R$.

Suppose there are no solutions to the equation $p(z) = 0$. Then

$$f(z) := \frac{1}{p(z)}$$

is entire. It follows that it is bounded on the closed and bounded set of point in the closed ball $\overline{B}$ of radius R centred at $z = 0$. In other words, there exists M_0 such that $|f(z)| \leq M_0$ for all z such that $|z| \leq R$. On the other hand, since $|p(z)| \geq 1$ when $|z| > R$, we have $|f(z)| \leq 1$ for z such that $|z| > R$. If we set $M = \max\{M_0, 1\}$, then we get

$$|f(z)| \leq M, \qquad z \in \mathbb{C}.$$

Thus f is bounded. By Louisville's theorem, f is a constant. This means p is a constant. However, the degree of p is n which is greater than 1, and so $p^{(n)}(z) = a_n \neq 0$, which means p is not a constant. This is a contradiction. Hence there exists a solution to the equation $p(z) = 0$.

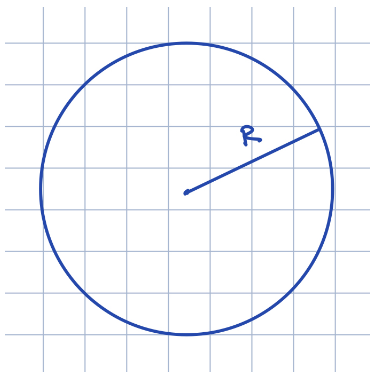

FIGURE 2. Outside the disc, $|p(z)| \geq 1$, and inside the disc, $|f(z)| \leq M_0$.

□

REFERENCES

[A] Lars V. Ahlfors, *Complex Analysis*, Third Edition, McGraw Hill, New York, 1979.