LECTURE 15

Date of Lecture: March 10, 2022

1. Using Cauchy's Integral Theorem and Integral Formula

1.1. **Recap of last lecture.** Here are the things we proved last time, which we will use.

I. (Cauchy's Integral Theorem) Let D be a simply connected region, Γ a loop in D, and f an analytic function on D. Then

$$\int_\Gamma f(z)dz = 0.$$

II. Let D be a simply connected domain. Every analytic function on D has an antiderivative.

III. Let Γ be a simple loop and f a function which is analytic on in the interior of Γ and at each point of Γ (i.e. analytic in a neighbourhood of each point of Γ), then

$$\int_\Gamma f(z)dz = 0.$$

This was the comment in 1.3.3 of Lecture 14. The point is that under the hypotheses, there is a simply connected domain D on which f us defined which contains Γ as well as its interior.

IV. Let Γ be a contour and f a continuous complex-valued function on Γ. Suppose M is a real number such that $|f(z)| \leq M$ for all $z \in \Gamma$. Then

$$\left|\int_\Gamma f(z)dz\right| \leq M\ell(\Gamma).$$

V. (Cauchy's Integral Formula) Let D be a simply connected domain, Γ a simple loop in D oriented in the positive direction and f an analytic function on D. Let Ω be the interior of Γ. Then

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}d\zeta, \qquad \text{for } z \in \Omega.$$

There is a minor variant of V, based on III above, namely:

VI. Let Γ be a simple loop oriented in the positive sense, Ω the interior of Γ, and f a function which is analytic in the interior Γ and in a neighbourhood of every point of Γ. Then

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}d\zeta, \qquad \text{for } z \in \Omega.$$

1.2. **Examples.**

1. Let Γ be simple loop, positively oriented, Ω the interior of Γ, and a a point in Ω. Then

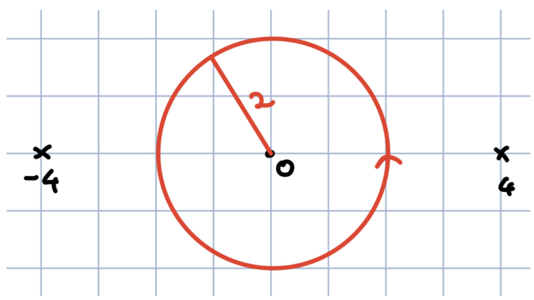
$$\int_\Gamma \frac{1}{z-a} dz = 2\pi i.$$

This follows from Cauchy's Integral Formula, i.e. V above, by taking $f(z) = 1$.

2. Let us evaluate

$$\int_\Gamma \frac{1}{z^2-16} dz$$

where Γ is the circle $|z| = 2$ oriented positively.



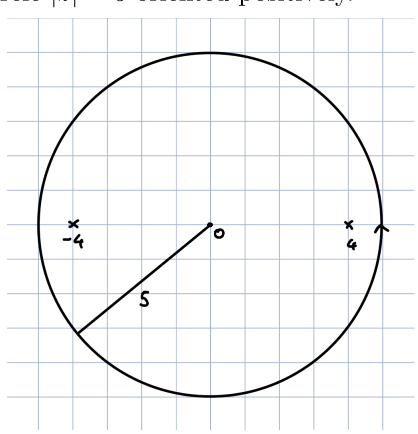
If $f(z) = \dfrac{1}{z^2-16}$, then f is analytic on $\mathbb{C} \setminus \{-4, 4\}$. In particular it is analytic in the interior of Γ as well as on every point of Γ. It follows that

$$\int_\Gamma \frac{1}{z^2-16} dz = 0.$$

3. Let us now integrate the integrand in the above example over a $|z| = 5$, i.e. let us evaluate

$$\int_\Gamma \frac{1}{z^2-16} dz$$

where now Γ is the circle $|z| = 5$ oriented positively.



This time the function f is no longer analytic in the interior of Γ since ± 4 lie inside Γ.

There are two ways of evaluating our integral. The first is to use the partial fraction decomposition

$$\frac{1}{z^2-16} = \frac{1}{8}\left(\frac{1}{z-4} - \frac{1}{z+4}\right).$$

Using the result in Example 1, we see that

$$\int_\Gamma \frac{1}{z^2-16}dz = \frac{1}{8}\int_\Gamma dz \frac{1}{z-4}dz - \int_\Gamma \frac{1}{z+4}dz = \frac{1}{8}(2\pi i) - \frac{1}{8}(2\pi i) = 0.$$

The second is to consider the two loops Γ_L and Γ_R. Γ_L (the "left loop") is the loop which is the left half of the given circle (traversed counterclockwise) plus the vertical line segment from $-5i$ to $5i$. Γ_R (the "right loop") is the loop which is the downward line segment from $5i$ to $-5i$, followed by the right half of the given circle oriented in the counter clockwise direction. It is clear that

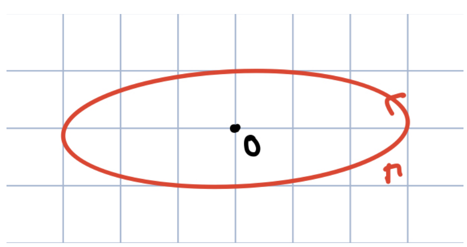
$$\begin{aligned}\int_\Gamma \frac{1}{z^2-16}dz &= \int_{\Gamma_L} \frac{1}{z^2-16}dz + \int_{\Gamma_R} \frac{1}{z^2-16}dz \\ &= \int_{\Gamma_L} \frac{1/(z-4)}{z+4}dz + \int_{\Gamma_R} \frac{1/(z+4)}{z-4}dz\end{aligned}$$

Now $f_L(z) = 1/(z-4)$ is analytic in the interior of Γ_L and on every point of Γ_L and hence (by the Cauchy Integral Formula) $\int_{\Gamma_L} f_L(z)/(z+4)dz = 2\pi i f_L(-4) = (2\pi i)(1/(-8)) = -(\pi i)/4$. Similarly, settling $f_R = 1/(z+4)$, one sees that $\int_{\Gamma_R} f_R(z)/(z-4)dz = 2\pi i f_R(4) = (2\pi i)(1/8) = (\pi i)/4$. Thus

$$\int_\Gamma \frac{1}{z^2-16}dz = \int_{\Gamma_L} \frac{f_L(z)}{z+4} + \int_{\Gamma_R} \frac{f_R(z)}{z-4}dz = -(\pi i)/4 + (\pi i)/4 = 0.$$

4. If Γ is the ellipse $x^2 + 4y^2 = 1$ traversed once in the positive sense, then clearly, by Example 1,

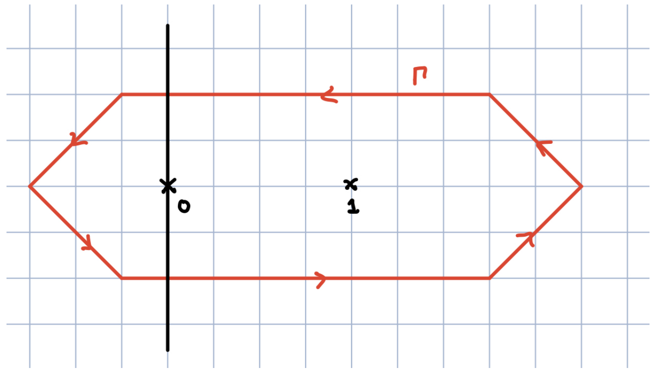
$$\int_\Gamma \frac{1}{z}dz = 2\pi i.$$



5. Let us evaluate

$$\int_\Gamma \frac{3z-2}{z^2-z}dz$$

where Γ is the curve in the picture below.



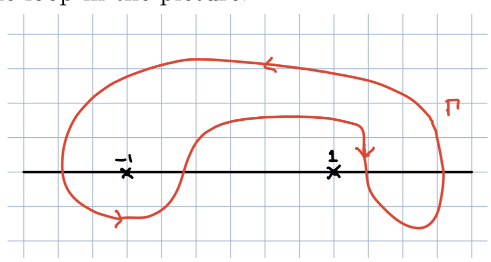
The trick is to use the partial fraction decomposition

$$\frac{3z-2}{z^2-z} = \frac{2}{z} + \frac{1}{z-1}.$$

Since $z = 0$ and $z = 1$ lie in the interior of Γ the formula in Example 1 gives

$$\int_\Gamma \frac{3z-2}{z^2-z}dz = \int_\Gamma \frac{2}{z}dz + \int_\Gamma \frac{1}{z-1}dz = 2(2\pi i) + 2\pi i = 6\pi i.$$

6. Let Γ be the simple loop in the picture.



Let us evaluate

$$\int_\Gamma \frac{1}{z^2-1}dz.$$

Once again there are two ways.
Method I: Let $f(z) = 1/(z-1)$. Then the required integral is $\int_\Gamma f(z)/(z+1)dz$. Now $f(z)$ is analytic in the interior of Γ and on Γ. So by the Cauchy Integral formula, we get

$$\int_\Gamma \frac{1}{z^2-1}dz = \int_\Gamma \frac{f(z)}{z+1} = 2\pi i f(-1) = 2\pi i(-\tfrac{1}{2}) = -\pi i.$$

Method II: Use the partial fraction decomposition,

$$\frac{1}{z^2-1} = \frac{1/2}{z-1} - \frac{1/2}{z+1}.$$

The integral of the first term in the partial fraction decomposition is zero by the Cauchy Integral Theorem since $1/(z-1)$ is analytic inside Γ and on Γ. The integral of the second term is $-(1/2)(2\pi i) = -\pi i$.

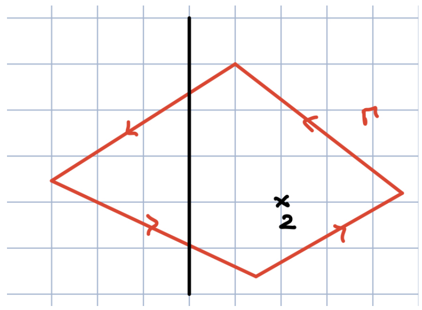
7. Now let us evaluate

$$\int_{|z-2|=3} \frac{e^z + \sin z}{z}dz,$$

where the circle $|z-2|=3$ is oriented in the counter-clockwise direction. It is clear that $z=0$ lies in the interior of the circle. If $f(z)=e^z+\sin z$ then, by the Cauchy Integral Formula,

$$\int\limits_{|z-2|=3} \frac{e^z+\sin z}{z}dz = 2\pi i f(0) = 2\pi i(e^0+\sin(0) = 2\pi i.$$

8. Let Γ be as in the picture.



Let us evaluate

$$\int_\Gamma \frac{\cos z}{z^2-4}dz.$$

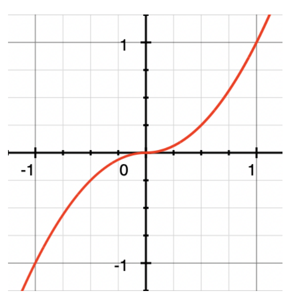
Let $f(z)=\cos z/(z+2)$. Note that f is analytic inside Γ and also at every point in Γ. The given integral can be re-written as i$\int_\Gamma f(z)/(z-2)dz$. Thus

$$\int_\Gamma \frac{\cos z}{z^2-4}dz = \int_\Gamma \frac{f(z)}{z-2}dz = 2\pi i f(2) = \frac{\pi i \cos(2)}{2}.$$

2. **Infinite differentiability of analytic functions**

2.1. **A real variable example.** Let $f \in \mathbb{R} \to \mathbb{R}$ be the function given by the rule:

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$$



One checks easily that f is differentiable for its right derivative at $x=0$ and its left derivative at $x=0$ are the same, and the two derivatives are zero at $x=0$. The picture above shows that the graph has a tangent at $x=0$. In fact

$$f'(x) = 2|x|, \qquad x \in \mathbb{R}.$$

Unfortunately, the first derivative f' is not differentiable at $x=0$ in this case. In other words $f''(0)$ does not exist.

This state of affairs is very different from what happens in complex analysis. An analytic function is in fact infinitely differentiable.

2.2. **Infinite differentiablility.** Let f be analytic on a domain D. If Γ is a simple loop in D whose interior Ω is in D, then the following formula holds.

$$f'(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{(\zeta - z)^2} dz, \qquad z \in \Omega. \tag{2.2.1}$$

In fact much more can be said. Namely that the n^{th} derivative $f^{(n)}$ of f exists on D, and within Ω (the interior of Γ) this n^{th} derivative is given by the formula

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} dz, \qquad z \in \Omega. \tag{2.2.2}$$

We gave a proof of (2.2.1) in class, but we will prove something more general in the next class, and so I will not give the proof here.