LECTURE 14

Date of Lecture: March 8, 2022

1. Integrals and simply connected regions

1.1. The main theorem in this section is the following:

Theorem 1.1.1. (Cauchy's Integral Theorem) *Let D be a simply connected region, Γ a loop in D, and f an analytic function on D. Then*

$$\int_\Gamma f(z)dz = 0.$$

Proof. Since D is simply connected, the loop Γ can be deformed to a point, and the integral over a point is zero. □

1.2. **The Green's Theorem approach.** In this subsection we will give a different proof of Theorem 1.1.1 than the one we above. This alternative proof uses Green's Theorem from vector calculus. Before we start on that we wish to state a purely topological fact.

Proposition 1.2.1. *A domain D is simply connected if for every simple loop Γ in D, the interior of Γ lies in D.*

We omit the proof of this topological fact. It is intuitively obvious. □

Proof of Theorem 1.1.1 using Green's Theorem. Suppose

$$\mathbf{F}(x,y) = V(x,y)\mathbf{i} + W(x,y)\mathbf{j}$$

is a vector field on D, with V and W having continuous first partial derivatives on D, and Γ is a simple loop in D oriented in the positive direction. Let R be the interior of Γ. Since D is simply connected, R lies inside D. Green's Theorem is the statement that the following equality holds

$$\int_\Gamma (V\,dx + W\,dy) = \iint_R \left(\frac{\partial W}{\partial x} - \frac{\partial V}{\partial y}\right) dA \tag{1.2.2}$$

The left side is the line integral $\int_\Gamma \mathbf{F}\cdot\mathbf{ds}$.

Coming to Theorem 1.1.1, suppose we write our analytic function f on D as $f = u + iv$ where u and v are the real and imaginary parts of f. We will assume that u and v have continuous first partial derivatives. (We assumed this in the earlier proof of the invariance of integrals too.) Let Γ be a simple loop and suppose it is oriented positively, and let R be its interior. By the definition given on page 4 of Lecture 12, we know $\int_\Gamma f(z)dz$ is equal to $\int_\Gamma (udx - vdy) + i\int_\Gamma (vdx + udy)$. This gives:

$$\begin{aligned}\int_\Gamma f(z)dz &= \int_\Gamma (udx - vdy) + i\int_\Gamma (vdx + udy)\\ &= \iint\limits_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dA + i\iint\limits_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dA\\ &= 0.\end{aligned}$$

The second equality is via (1.2.2) and the third via the Cauchy-Riemann equations. If Γ is negatively oriented then $-\Gamma$ is positively oriented and since $\int_\Gamma = -\int_{-\Gamma}$ we see that in this case too $\int_\Gamma f(z)dz = 0$. Next if Γ is not a simple loop, it can be broken up into a finite number of simple loops in D and so once again, we see that $\int_\Gamma f(z)dz = 0$.

This completes the alternative proof of Theorem 1.1.1 via Green's Theorem. □

1.3. **Antiderivatives and simply connected regions.** Recall that we proved that a continuous complex valued function f on a domain D has an antiderivative on D if and only if $\int_\Gamma f(z)dz = 0$ for all loops in D. An immediate corollary of this and the Cauchy Integral Theorem (i.e. Theorem 1.1.1) is

Theorem 1.3.1. *Let D be a simply connected domain. Every analytic function on D has an antiderivative.*

Examples 1.3.2. Here are some examples illustrating some of the results discussed above.

1. We know that $\int_C z^{-1}dz = 2\pi i \neq 0$, where C is the unit circle $|z| = 1$ oriented in the positive direction. This proves, by Theorem 1.1.1, that $\mathbb{C} \setminus \{0\}$ is not simply connected, for z^{-1} is an analytic function on $\mathbb{C} \setminus \{0\}$. It also shows that $1/z$ has no antiderivative on $\mathbb{C} \setminus \{0\}$. Figure 1 may provide a more intuitive explanation of reason the punctured plane $\mathbb{C} \setminus \{0\}$ is not simply connected.

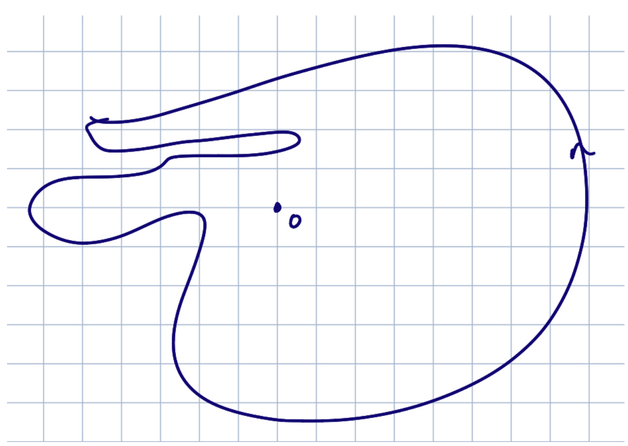

Figure 1. The curve shown cannot be shrunk to a point in the punctured plane $\mathbb{C} \setminus \{0\}$.

2. The set $D = \mathbb{C} \smallsetminus (-\infty, 0]$ is simply connected. Indeed, let Γ be a loop in D, say with parameterization $\zeta(t)$, $t \in [0, 1]$. Set $z(s, t) = s + (1 - s)\zeta(t)$ for $(s, t) \in [0, 1] \times [0, 1]$. It is easy to check that $z(s,t)$ lies in D for every $(s, t) \in [0, 1] \times [0, 1]$. In fact $z(s, t)$ lies in the line segment joining $\zeta(t)$ to $z = 1$ (see FIGURE 2). Moreover, it gives a deformation of Γ to the point 1.

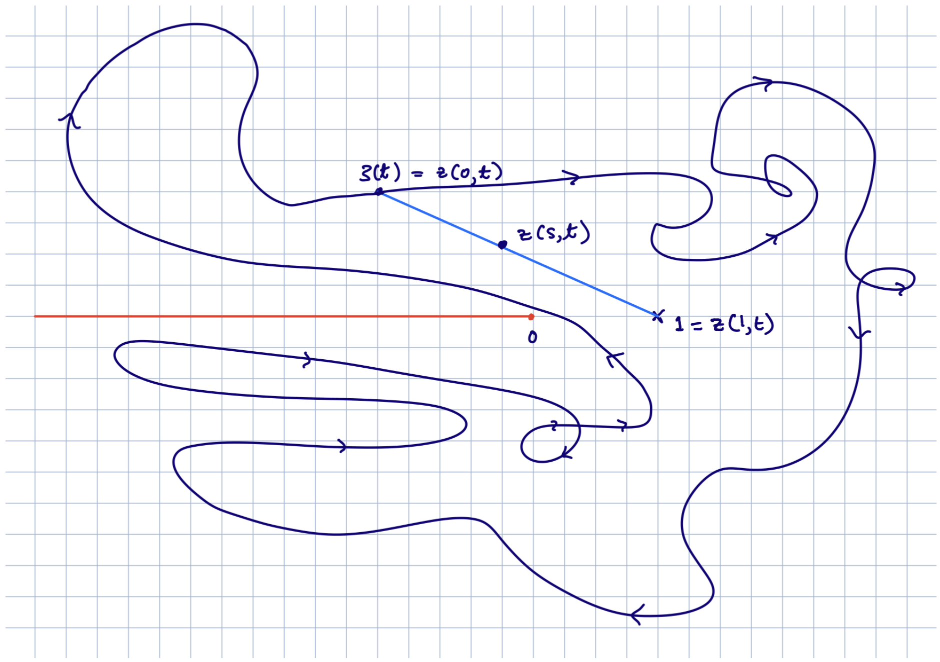

FIGURE 2. For $s \in [0, 1]$, $z(s,t) = s + (1 - s)\zeta(t)$ lies in the line segment joining $\zeta(t)$ to 1, and so lies in $\mathbb{C} \smallsetminus (-\infty, 0]$.

By Theorem 1.3.1 $1/z$ must have an antiderivative on D. Two antiderivatives of $1/z$ differ by a constant, and if we know the value of the antiderivative at any one point in D, we know it everywhere. The antiderivative which is zero at $z = 1$ is the principal logarithm Log.

1.3.3. Suppose Γ is a simple loop, Ω its interior, and f a function which is analytic in an open set U which contains $\Omega \cup \Gamma$. Then there is simply connected domain D which is sandwiched between $\Omega \cup \Gamma$ and U: $\Omega \cup \Gamma \subset D \subset U$. An immediate consequence is this: If one has a function f which is analytic on Ω and at each point of Γ (i.e. analytic in a neighbourhood od each point of Γ), then

$$\int_\Gamma f(z)dz = 0.$$

The proof is that, we can find a simply connected domain D containing Γ, and Cauchy's Integral formula applies to all loops in D.

2. A fundamental inequality

2.1. **The absolute value of an integral.** Let $g\colon [a, b] \to \mathbb{C}$ be a continuous function. We know that $\int_a^b g(t)dt$ is the limit of sums of the form $\sum_i g(t_i^*)(t_i - t_{i-1})$

where

$$\wp : \; a = t_0 < t_1 < \cdots < t_n = b$$

is a partition of the interval $[a, b]$ into subintervals, and t_i^* is any point in $[t_{i-1}, t_i]$. More precisely, if $\mu(\wp)$ is the maximum length of the subintervals in $\wp$, then

$$\int_a^b g(t)dt = \lim_{\mu(\wp)\to 0} \sum_{i=1}^n g(t_i^*)(t_i - t_{i-1}).$$

Now $|\sum_{i=1}^n g(t_i^*)(t_i - t_{i-1})| \le \sum_{i=1}^n |g(t_i^*)(t_i - t_{i-1})|$ by the triangle inequality. Taking limits as $\mu(\wp) \to 0$, we get

$$\left| \int_a^b g(t)dt \right| \le \int_a^b |g(t)|dt. \tag{2.1.1}$$

2.2. **The length of a contour.** Let $z(t)$, $t \in [a, b]$ be a parameterization of a contour Γ. Suppose we write $z(t) = x(t) + iy(t)$. From vector calculus we know that the length $\ell(\Gamma)$ of Γ is given by the formula

$$\ell(\Gamma) = \int_a^b \left\{ \frac{dx(t)}{dt}^2 + \frac{dy(t)}{dt}^2 \right\}^{1/2} dt.$$

Since $|z'(t)| = \sqrt{\frac{dx(t)}{dt}^2 + \frac{dy(t)}{dt}^2}$, we get

$$\ell(\Gamma) = \int_a^b |z'(t)|dt. \tag{2.2.1}$$

The following is a crucial result

Theorem 2.2.2. *Let Γ be a contour and f a continuous complex-valued function on Γ. Suppose M is a real number such that $|f(z)| \le M$ for all $z \in \Gamma$. Then*

$$\left| \int_\Gamma f(z)dz \right| \le M\ell(\Gamma).$$

Proof. Let $z(t)$, $t \in [a,\, b]$ be a parameterization of Γ. Then

$$\begin{aligned} \left| \int_\Gamma f(z)dz \right| &= \left| \int_a^b f(z(t))z'(t)dt \right| \\ &\le \int_a^b |f(z(t))||z'(t)|dt \qquad \text{(by (2.1.1))} \\ &\le M \int_a^b |z'(t)|dt \\ &= M\ell(\Gamma) \qquad \text{(by (2.2.1))} \end{aligned}$$

This is the inequality we had to establish. □

3. The Cauchy Integral Formula

3.1. Let D be a simply connected domain, Γ a simple loop in D oriented in the positive direction, Ω the interior of Γ. Since D is simply connected, Ω is a subset of D. Next let z_0 be a point in Ω. We can find a positive number r small enough that the closed disc $\{z \mid |z - z_0| \le r\}$ of radius r centred at z_0 lies entirely in Ω.

Let C_r be circle $|z - z_0| = r$ oriented positively. It is not hard to see that if h is an analytic function on $D \smallsetminus \{z_0\}$ then

$$\int_\Gamma h(z)dz = \int_{C_r} h(z)dz \tag{3.1.1}$$

This can be seen in two ways. Either note that Γ can be deformed to C_r in $D \smallsetminus \{z_0\}$. This is a topological fact which is intuitively clear. See, for example, FIGURE 3.

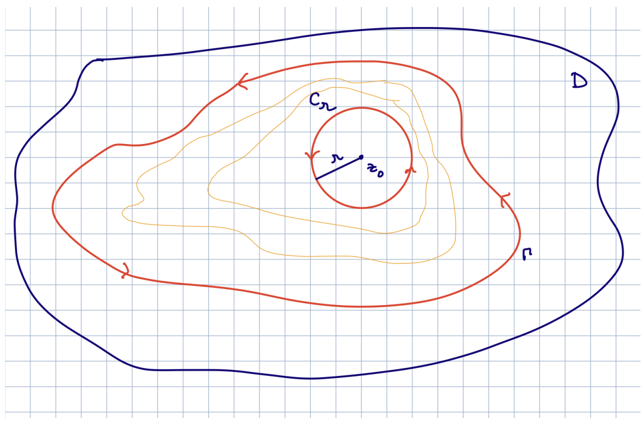

FIGURE 3. Γ and C_r are deformable to each other in $D \smallsetminus \{z_0\}$.

Another way is to break up the area between the two curves into two simply connected regions.

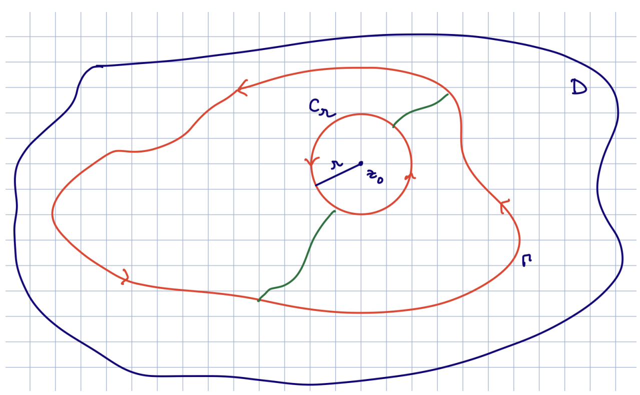

FIGURE 4. Breaking up the region between Γ and C_r into two pieces.

Now let Γ_L be the left loop determined by the "left sides" of Γ and C_r and the two green curves above.

Similarly one has the right loop Γ_R.

Since Γ_L and Γ_R are simple loops and h is analytic in their interiors and on them, $\int_{\Gamma_L} h(z)dz = \int_{\Gamma_R} h(z)dz = 0$ (see 1.3.3). It is easy to see that we have the

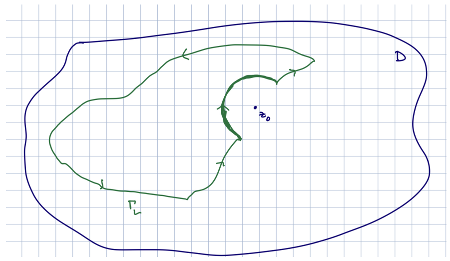

FIGURE 5. The contour Γ_L.

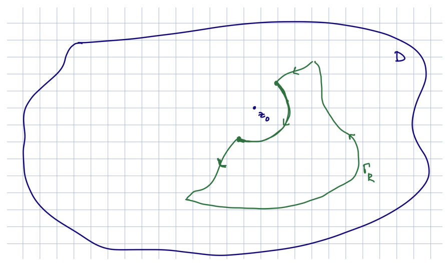

FIGURE 6. The contour Γ_R.

following relationship: $\int_{\Gamma_L} h(z)dz + \int_{\Gamma_R} h(z)dz = \int_\Gamma h(z)dz - \int_{C_r} h(z)dz$. From this it follows that $\int_\Gamma h(z)dz - \int_{C_r} h(z)dz = 0$, proving (3.1.1)

Now suppose f is analytic on D. Define a new function $g\colon D \setminus \{z_0\}$ by the rule:

$$g(z) = \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} & \text{when } z \in D \setminus \{z_0\} \\ \\ f'(z_0) & \text{when } z = z_0 \end{cases} \tag{3.1.2}$$

It is clear that g is analytic on $D \setminus \{z_0\}$. Moreover,

$$\lim_{z \to z_0} g(z) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) = g(z_0).$$

Thus g is continuous at z_0. Thus g is continuous on D, in particular on the set $\Omega \cup \Gamma$. Now $\Omega \cup \Gamma$ is a closed and bounded set, and so every continuous real function on it is bounded, and hence $|g|$ is bounded on $\Omega \cup \Gamma$. In other words we have a positive number M such that

$$|g(z)| \leq M, \qquad \forall z \in \Omega \cup \Gamma. \tag{3.1.3}$$

Since g is analytic on $D \setminus \{z_0\}$, by (3.1.1) we have $\int_\Gamma g(z)dz = \int_{C_r} g(z)dz$. It follows that

$$\left| \int_\Gamma g(z)dz \right| = \left| \int_{C_r} g(z)dz \right| \leq M\ell(C_r) = M(2\pi r).$$

Letting $r \downarrow 0$, we see that $\int_\Gamma g(z)dz = 0$. In other words, using the definition of g in (3.1.2) we have

$$\int_\Gamma \frac{f(z) - f(z_0)}{z - z_0} dz = 0.$$

This means

$$\int_\Gamma \frac{f(z)}{z-z_0}dz = \int_\Gamma \frac{f(z_0)}{z-z_0}dz = f(z_0)\int_\Gamma \frac{1}{z-z_0}dz.$$

Once again using (3.1.1), with $h(z) = 1/(z-z_0)$, we get (using the parameterization $z(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$ for C_r):

$$\int_\Gamma \frac{f(z)}{z-z_0}dz = f(z_0)\int_\Gamma \frac{1}{z-z_0}dz = f(z_0)\int_{C_r} \frac{1}{z-z_0}dz = (2\pi i)f(z_0),$$

In other words

$$f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-z_0}dz. \tag{3.1.4}$$

We re-state the above in the following way

Theorem 3.1.5. (Cauchy Integral Formula) *Let D be a simply connected domain, Γ a simple loop in D oriented in the positive direction and f an analytic function on D. Let Ω be the interior of Γ. Then*

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}d\zeta, \qquad \textit{for } z \in \Omega.$$

Proof. This is just a re-statement of (3.1.4). □