LECTURE 13

Date of Lecture: March 3, 2022

1. Loops and the Jordan curve theorem

1.1. **Loops.** Recall that a contour Γ is either a single point z_0 or a finite sequence of directed smooth curves $(\gamma_1, \ldots, \gamma_n)$ such that the terminal point of γ_k is the initial point of γ_{k+1} for $k = 1, \ldots, n$. We often write $\Gamma = \gamma_1 + \cdots + \gamma_n$ rather than $\Gamma = (\gamma_1, \ldots, \gamma_n)$.

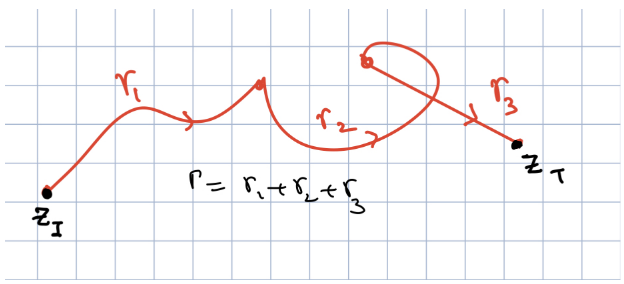

FIGURE 1. A contour. The initial point is usually denoted z_I and the terminal point z_T

Γ is said to be a *closed contour* or a *loop* if its initial and terminal points coincide. A *simple closed contour* or a *simple loop* is a loop which has no multiple points other than its initial (which is also its terminal) point. We also use the term loop or closed curve for undirected curves whose initial and terminal points are the same.

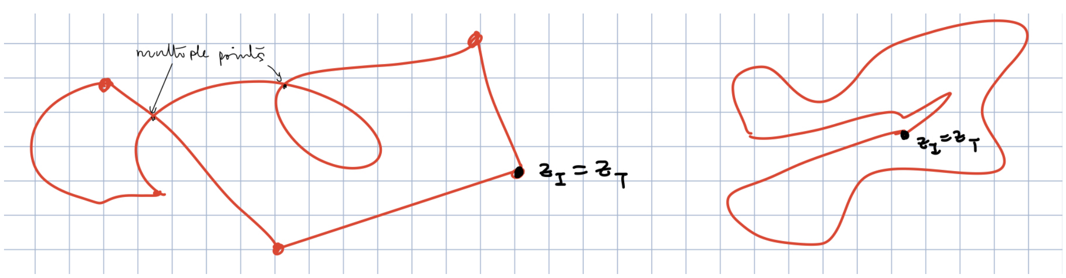

FIGURE 2. Two loops. The one on the left is not a simple loop, while the one on the right is.

The main theorem concerning simple loops is *Jordan's Curve Theorem* which is the following theorem:

Theorem 1.1.1. *A simple loop separates the complex plane into two domains, one bounded and the other unbounded, each having the loop as its boundary. The bounded domain is called the* interior *of the loop and the unbounded domain the* exterior.

We can use the Jordan curve theorem to define the positive orientation of a simple loop. If the simple loop Γ is directed in such a way that the interior lies to the left as one travels in the direction of the directed loop, then Γ is said to be *positively oriented*. Otherwise (i.e. when the interior falls to the left as one traverses the directed loop), Γ is said to be *negatively oriented*.

We will not be supplying a proof of this theorem, since it is an advanced theorem beyond the scope of this course.

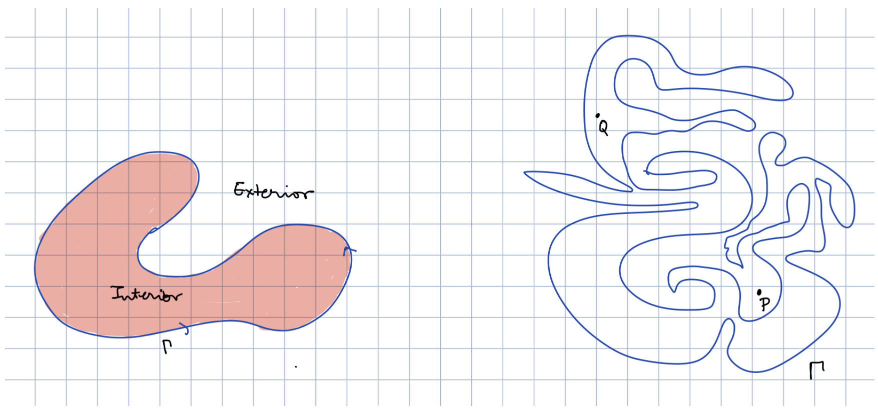

FIGURE 3. The simple loop on the left is positively oriented. The loop on the right is also simple, but working out the interior and exterior is a little more complicated. Check that the point P is in the exterior and the point Q is in the interior.

2. Path independence

2.1. Path independence and antiderivatives. In Lecture 7 (see page 6 of that lecture), we proved that if f is a continuous complex-valued function on a domain D such that f has an antiderivative F on D (i.e. $F'(z) = f(z)$), then for any contour Γ in D with initial point z_I and terminal point z_T we have the formula

$$\int_\Gamma f(z)dz = F(z_T) - F(z_I). \tag{2.1.1}$$

In other words, the integral $\int_\Gamma f(z)dz$ is *path independent*, i.e. it only depends upon f and the end points of Γ and not on Γ itself. Here is the Theorem we are interested in.

Theorem 2.1.2. *Let f be continuous on a domain D. The following are equivalent.*

(a) *f has an antiderivative on D.*
(b) *$\int_\Gamma f(z)dz = 0$ for every loop Γ in D.*
(c) *The contour integrals of f are independent of path, i.e. if Γ_1 and Γ_2 are two contours in D with the same initial points and the same terminal points, then $\int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz$.*

Proof. Assume (a). Then (b) must be true by formula (2.1.1), since for a loop Γ, $z_I = z_T$.

Now assume (b) is true. Suppose Γ_1 and Γ_2 have the same initial and terminal points. Then $\Gamma_1 - \Gamma_2$ is a loop, and so by (b), $\int_\Gamma f(z)dz = 0$. This means $\int_{\Gamma_1} f(z)dz - \int_{\Gamma_2} f(z)dz = 0$, giving (c).

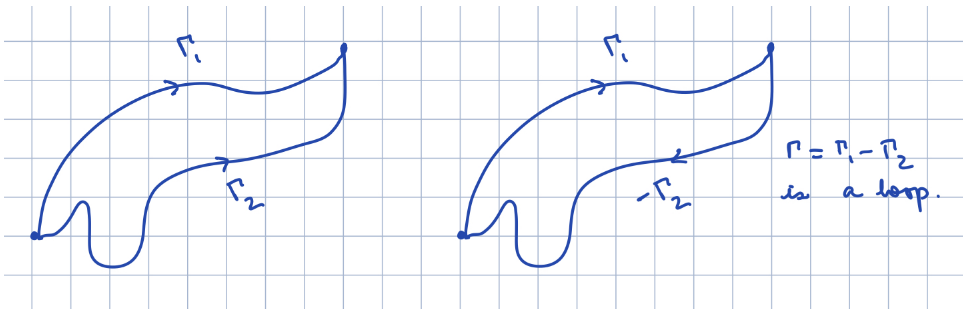

FIGURE 4. $\Gamma = \Gamma_1 - \Gamma_2$ is a loop.

We will now assume (c) and prove (a). Pick a point z_0 in D and fix it. Since D is a domain, it is connected. Let z be a point in D. Since D is connected, we have a contour Γ in D starting at z_0 and terminating at z. Define

$$F(z) = \int_\Gamma f(z)dz.$$

By our assumption (c), the above integral does not depend on the contour Γ, so long as it starts at z_0 and terminates at z.

Let B be a small circular neighbourhood of z in D (since D is open, it is always possible to find such a B). Let Δz be such that $|\Delta z|$ is small enough that $z + \Delta z$ lies in B. Let Γ_1 be the line segment joining z to $z + \Delta z$. Then $\Gamma + \Gamma_1$ is a contour in D whose initial point is z_0 and terminal point is $z + \Delta z$. (See FIGURE 5.)

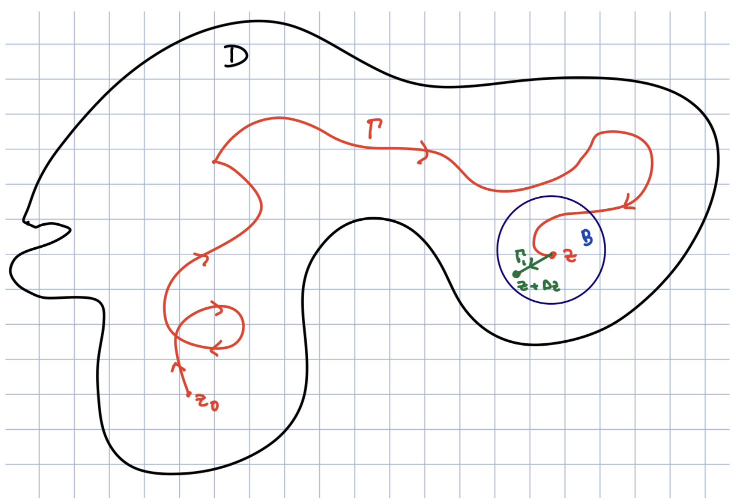

FIGURE 5.

By definition of F this means

$$F(z+\Delta z) = \int_\Gamma f(z)dz + \int_{\Gamma_1} f(z)dz = F(z) + \int_{\Gamma_1} f(z)dz.$$

It follows that

$$\frac{F(z+\Delta z)-F(z)}{\Delta z} = \frac{\int_{\Gamma_1} f(z)dz}{\Delta z}.$$

Now Γ_1 can be parameterized as: $z(t) = z + t\Delta z$, $0 \le t \le 1$. Thus

$$\frac{F(z+\Delta z)-F(z)}{\Delta z} = \frac{\int_0^1 f(z+t\Delta z)(\Delta z)dt}{\Delta z} = \int_0^1 f(z+t\Delta z)dt.$$

Since f is continuous, $f(z+t\Delta z) \approx f(z)$ for Δz such that $|\Delta z|$ is small, where the symbol $\approx$ is for "approximately". In fact, $\lim_{\Delta z\to 0} f(z+\Delta z) = f(z)$. This means that if $|\Delta z|$ is small, then

$$\frac{F(z+\Delta z)-F(z)}{\Delta z} \approx \int_0^1 f(z)dt = f(z)\int_0^1 dt = f(z).$$

Letting $\Delta z \to 0$ we get

$$\lim_{\Delta z\to 0} \frac{F(z+\Delta z)-F(z)}{\Delta z} = f(z).$$

Thus F is differentiable at all points in D and $F'(z) = f(z)$. This proves (a), assuming (c). $\square$

2.1.3. Remark. The argument given towards the end of the above proof can be made rigorous using an ε-δ argument. By continuity of f we know that for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(z+h) - f(z)| < \varepsilon$ whenever $|h| < \delta$. Now if $|\Delta z| < \delta$, then $|t\Delta z| < \delta$ for every $0 \le t \le 1$. It follows that $|f(z+t\Delta z) - f(z)| < \varepsilon$ whenever $|Dez| < \delta$ and $0 \le t \le 1$. How do we proceed from here? Think it through and see if you can show that $|\int_0^1 f(z+t\Delta z)dt - \int_0^1 f(z)dt| < \varepsilon$ whenever $|\Delta z| < \delta$. Note that the variable of integration in both the integrals above is t and not z. After you show that, how will you show that $F' = f$?

Examples 2.1.4. Here are some examples illustrating the path independence theorem.

1. Let D be the complex plane punctured at the origin, i.e. $D = \mathbb{C} \setminus \{0\}$. Let C be the circle $|z| = 1$ oriented in a positively.

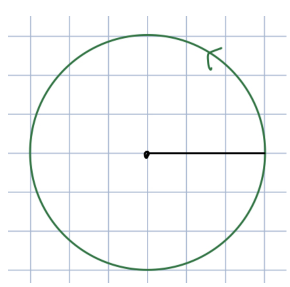

FIGURE 6.

We know that

$$\int_C \frac{1}{z} dz = 2\pi i \neq 0.$$

So by Theorem 2.1.2, $1/z$ does not have an antiderivative on D.

2. The same reasoning can be applied to the second problem in Homework 6 (this is problem 14 from section 3.3 of the text).

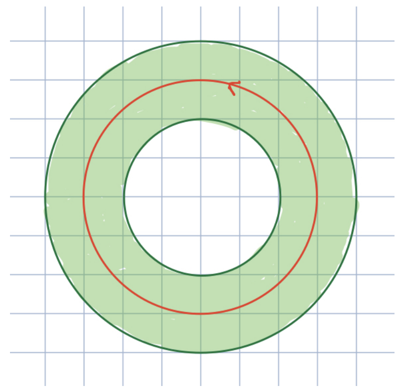

FIGURE 7. The annulus $D = \{z \mid 1 < |z| < 2\}$. The red oriented circle is $|z| = 3/2$ oriented positively.
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Let Γ be the circle $|z| = \frac{3}{2}$ oriented positively. Then Γ lies in the region D given by $1 < |z| < 2$ (see FIGURE 7). The integral $\int_\Gamma z^{-1} dz = 2\pi i \neq 0$, and so by Theorem 2.1.2, $1/z$ does not have an antiderivative on D.

3. Deformation of loops

3.1. **Intuitive ideas.** Suppose R and r are two positive real numbers with $R > r$. Let Γ_0 be the circle $|z| = R$, and Γ_1 the circle $|z| = r$, both oriented positively.

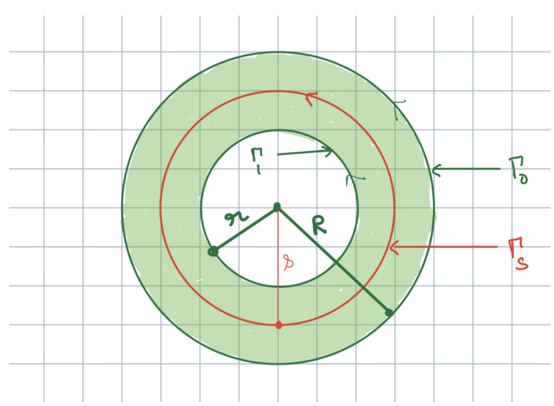

FIGURE 8. Γ_0 is the outer circle, Γ_1 is the inner circle, and Γ_s a "variable circle" between Γ_0 and Γ_1.

From FIGURE 8 it seems intuitive that we can "shrink" or "deform" Γ_0 in some continuous fashion to Γ_1 with the intermediate stages of the "deformation" being circles centred at $z = 0$ of radii between r and R. In fact for $0 \le s \le 1$ we can find intermediate circles Γ_s as in the picture which in mathematical way captures this deformation process. For $s \in [0, 1]$, define, as Andy Hsiao suggested in class, Γ_s to be the circle $|z| = sr + (1 - s)R$ oriented in the positive direction. Let

$$z(s, t) = \{sr + (1 - s)R\}e^{i2\pi t}, \qquad 0 \le t \le 1.$$

Then $t \mapsto z(s, t)$, $0 \le t \le 1$ is a parameterization of Γ_s for $0 \le s \le 1$. Note that as $s \to 0$ the red positively oriented circle Γ_s in the picture approaches the oriented circle Γ_0, and as $s \to 1$, Γ_s approach the oriented circle Γ_1.

Similarly, if D is the complex plane, it is intuitively clear that we can shrink Γ_1 to a point, namely $z = 0$. A little more mathematically, if for each $s \in [0,\ 1]$ and each $t \in [0,\ 1]$ we set $z(s, t) = se^{i2\pi t}$, then for each fixed $0 < s \le 1$, the mapping $t \mapsto z(s,\ t)$ gives us gives us the circle $|z| = s$ oriented in the positive direction. Call this oriented circle Γ_s. When $s = 0$, we just get the origin $z = 0$. As $s \downarrow 0$ we see that $\Gamma_s \to \{0\}$. In other words the unit circle centred at $z = 0$ can be deformed to its centre. FIGURE 9 below may help visualise the situation.

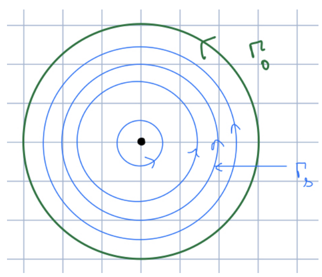

FIGURE 9. Deforming the circle of radius 1 centred at $z = 0$ to the origin in the complex plane.

3.2. **Deformations.** Roughly, in accordance with the examples in § 3.1, a deformation of a loop Γ_0 to a loop Γ_1 in a domain D looks like this picture.

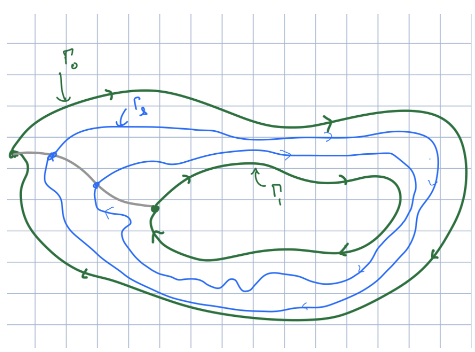

FIGURE 10. Deformation of Γ_0 to Γ_1 through intermediary loops Γ_s.

This motivates the following more rigorous definition:

Definition 3.2.1. Let Γ_0 and Γ_1 be loops in a region D in the complex plane. Γ_0 is said to be *continuously deformable to* Γ_1 if there is a continuous function

$$z\colon [0,\,1]\times[0,\,1] \longrightarrow D; \qquad (s,t)\mapsto z(s,\,t)$$

such that

(1) For each fixed $s\in[0,\,1]$, the function $t\mapsto z(s,\,t)$, $0\le t\le 1$, parameterizes a loop Γ_s in D;
(2) $t\mapsto z(0,\,t)$, $0\le t\le 1$, parameterizes Γ_0;
(3) $t\mapsto z(1,\,t)$, $0\le t\le 1$, parameterizes Γ_1.

In the above situation we also also say that Γ_1 *is a deformation of* Γ_0 *in* D. Sometimes the function $(s,\,t)\mapsto z(s,\,t)$ is called a *deformation of* Γ_0 *to* Γ_1 *in* D.

3.2.2. Let Γ_0, Γ_1 and Γ_2 be three loops in a domain D. The set $[0,\,1]\times[0,\,1]$ is called the *closed unit square* or just the *unit square* in $\mathbb{C}$. Here are three basic observations.

1. Clearly Γ_0 is a deformation of itself. Indeed if $\zeta(t)$, $0\le t\le 1$ is a paramemetization of Γ_0, then take $z(s,\,t)$ to be the function given by $z(s,\,t)=\zeta(t)$ on the unit square.

2. If Γ_1 is a deformation of Γ_0 in D, then Γ_0 is a deformation of Γ_1 in D. To see this, suppose $z(s,\,t)$ in the continuous function on the unit square which gives the deformation. Then the function ζ on the unit square given by $\zeta(s,\,t)=z(1-s,\,t)$ gives the deformation of Γ_1 to Γ_0.

3. Suppose Γ_1 is a deformation of Γ_0 in D, say via a continuous function z_1 on the unit square, and Γ_2 is a deformation of Γ_1 in D, say via a continuous function z_2 on the unit square, then define $(s,\,t)\mapsto z(s,\,t)$ on the unit square as follows:

$$z(s,\,t)=\begin{cases} z_1(2s,\,t) & \text{when } (s,\,t)\in[0,\,1/2]\times[0,\,1];\\ z_2(2s-1,t) & \text{when } (s,\,t)\in[1/2,\,1]\times[0,\,1].\end{cases}$$

It is easy to check that that $z(s,\,t)$ gives a deformation of Γ_0 to Γ_2.

3.3. **Simply connected regions.** A domain D in the complex plane is called *simply connected* if every loop in D can be deformed to a point. (The intermediary loops Γ_s are also required to be in D.) A domain which is not simply connected is often called a *mulitiply connected* domain,

Examples 3.3.1. Here are some examples of simply connected and multiply connected domains

1. The complex plane $\mathbb{C}$ is simply connected. To see this, let Γ_0 be any loop, and let $z(t)$, $0\le t\le 1$ be a parameterization of Γ_0. Set

$$z(s,\,t)=(1-s)z(t), \qquad (s,\,t)\in[0,\,1]\times[0,\,1].$$

I leave it to you to check that this gives a deformation of Γ_0 to a point.

2. The open disc $B_r(z_0)$ of radius r centred at a point z_0 is also simply connected, by essentially the same argument as above. Suppose Γ_0 is a loop in $B_r(z_0)$, and let $z(t)$, $0\le t\le 1$ be a parameterization of Γ_0. Set $z(s,t)=z_0+(1-s)(z(t)-z_0)$ for $(s,\,t)\in[0,\,1]\times[0,\,1]$. If $(s,\,t)\in[0,\,1]\times[0,\,1]$, we have $|z(s,\,t)-z_0| = (1-s)|z(t)-z_0|\le|z(t)-z_0|<r$. We have just proved that $z(s,\,t)$ lies in the open disc $B_r(z_0)$ for every $(s,\,t)$ in the unit square $[0,\,1]\times[0,\,1]$. It is now clear

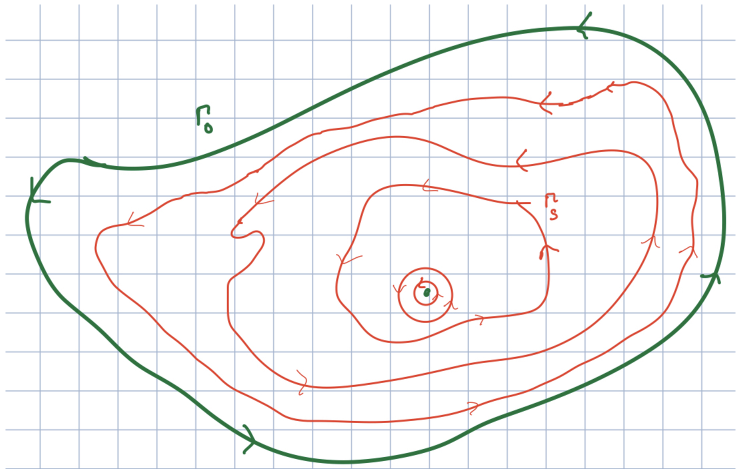

FIGURE 11. The loop Γ_0 can be deformed to a point in a simply connected domain.

that $(s, t) \mapsto z(s, t)$ gives a deformation of Γ_0 to the point z_0.

3. The annulus in FIGURE 7 is not simply connected. The red circle shown in the figure cannot be deformed within the annulus to a point.

3.4. **The Deformation Ivariance Theorem.** Here is the main theorem concerning deformations and contour integrals of analytic functions.

Theorem 3.4.1. *Let Γ_0 and Γ_1 be loops in a domain D such that Γ_1 is a deformation of Γ_0 in D. Then for every analytic function f on D we have*

$$\int_{\Gamma_0} f(z)dz = \int_{\Gamma_1} f(z)dz.$$

Proof. Let f be analytic on D. We give a weak version of the proof. We will assume two things

(i) The function $(s, t) \to z(s, t)$ giving the continuous deformation from Γ_0 to Γ_1 has continuous partial derivatives with respect to s and t. This has an important consequence, namely

$$\frac{\partial^2 z}{\partial s \partial t}(s, t) = \frac{\partial^2 z}{\partial t \partial s}(s, t).$$

(ii) We will also assume that the derivative f' is continuous.

The assumption in (i) can be dropped, but the proof requires techniques from topology and analysis beyond the scope of this course. As for the assumption in (ii), in fact we do not need to assume this, because a famous result of Goursat shows that the derivative of an analytic function is always continuous. However, once again, the proof of Goursat's result is advanced and beyond the scope of this course.

Fix $s \in [0, 1]$. Set $I_s = \int_{\Gamma_s} f(z)dz$. Note that

$$I_s = \int_0^1 f(z(s, t)\frac{\partial z}{\partial t}(s, t)dt.$$

We have to show that $I_0 = I_1$. We will do this by showing that $dI_s/ds = 0$ for $s \in [0,\, 1]$, so that $s \mapsto I_s$ is a constant, whence $I_0 = I_1$.

Under our assumptions (i) and (ii), the integrand $f(z(s,\, t)\frac{\partial z}{\partial t}(s,\, t)$ is continuously differentiable in s and this condition allows us the differentiate I_s under the integral sign with respect to s. Thus we have:

$$\begin{aligned}\frac{dI_s}{ds} &= \frac{d}{ds}\int_0^1 f(z(s,\, t))\frac{\partial z}{\partial t}(s,\, t)dt \\ &= \int_0^1 \frac{\partial}{\partial s}\Big(f(z(s,\, t))\frac{\partial z}{\partial t}(s,\, t)\Big)dt \\ &= \int_0^1 \Big(f'(z(s,\, t))\frac{\partial z}{\partial s}(s,\, t)\frac{\partial z}{\partial t}(s,\, t) + f(z,\, t)\frac{\partial^2 z}{\partial s \partial t}(s,\, t)\Big)dt\end{aligned}$$

Now, under our assumptions, the mixed partials $\frac{\partial^2 z}{\partial s \partial t}(s,\, t)$ and $\frac{\partial^2 z}{\partial t \partial s}(s,\, t)$ are equal, and hence

$$f'(z(s,\, t))\frac{\partial z}{\partial s}(s,\, t)\frac{\partial z}{\partial t}(s,\, t) + f(z,\, t)\frac{\partial^2 z}{\partial s \partial t}(s,\, t) = \frac{\partial}{\partial t}\Big(f(z(s,\, t))\frac{\partial z}{\partial s}(s,\, t)\Big).$$

This means

$$\begin{aligned}\frac{dI_s}{ds} &= \int_0^1 \frac{\partial}{\partial t}\Big(f(z(s,\, t))\frac{\partial z}{\partial s}(s,\, t)\Big)dt \\ &= f(z(s,\, 1))\frac{\partial z}{\partial s}(s,\, 1) - f(z(s,\, 0))\frac{\partial z}{\partial s}(s,\, 0).\end{aligned}$$

Now, $z(s,\, 1) = z(s,\, 0)$ for all s in $[0,\, 1]$, and hence $f(z(s,\, 1)) = f(z(s,\, 0))$ and $\frac{\partial z}{\partial s}(s,\, 1) = \frac{\partial z}{\partial s}(z,\, 0)$. This proves that I_s is identically zero for $s \in [0,\, 1]$, showing that $I_0 = I_1$, as required. $\square$