Formal definitions:

A point set γ is said to be a smooth arc if it is the range of some continuous complex-valued function $z = z(t)$, $a \le t \le b$, that satisfies the following conditions (with $z(t) = x(t) + iy(t)$ being the decomposition into real and imaginary parts):

(i) $z(t)$ has a continuous derivative on $[a,b]$ (with $z'(a)$ being the left derivative and $z'(b)$ being the right derivative)

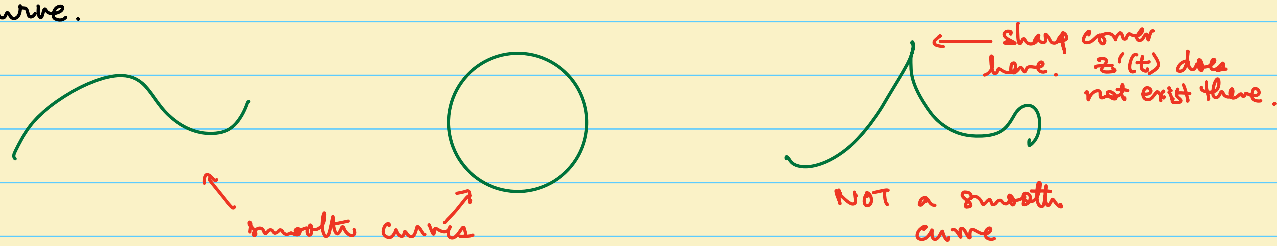
(ii) $z'(t) = x'(t) + iy'(t)$ never vanishes on $[a,b]$

(iii) $z(t)$ is one-to-one on $[a,b]$, i.e. if $a \le t, s \le b$, $t \ne s$, then $z(t) \ne z(s)$.

A point set is called a smooth closed curve if it is the range of some continuous function $z = z(t)$, $a \le t \le b$, satisfying conditions (i) and (ii) and the following

(iii)' $z(t)$ is one-to-one on the half-open interval $[a,b)$, but $z(b) = z(a)$ and $z'(b) = z'(a)$. ← This condition is often omitted but we will keep it.

A smooth curve is either a smooth arc or a smooth closed curve.



Admissible parameterizations: A smooth curve may have more than one parameterization. A parameterization which satisfies (i), (ii), (iii) or (i), (ii), (iii)' above is called an admissible parameterization.

Example: Consider the curve γ which is the graph of $y = x^3$ in the interval $-1 \le x \le 1$. Then

$$z(t) = t^3 + it^9, \quad -1 \le t \le 1$$

is a parameterization, but it is not an admissible parameterization since $z'(0) = 0$. However

$$z(t) = t + it^3, \quad -1 \le t \le 1$$

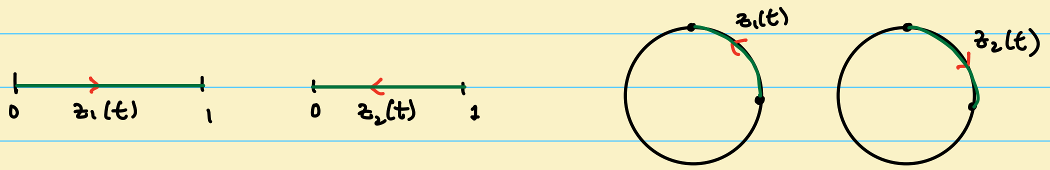
is an admissible parameterization of γ.

Question: Is $z(t) = \sin(t) + i\sin^3(t)$, $-\frac{\pi}{2} \le t \le \pi/2$ an admissible parameterization of the above γ? What about $z(t) = 3t+4 + i(3t+4)^3$, $-5/3 \le t \le -1$?

Directed curves:

Examples: Let γ be the line segment from 0 to 1 on the real axis. Then $z_1(t) = t$, $0 \le t \le 1$ and $z_2(t) = 1-t$, $0 \le t \le 1$ are two admissible parameterizations of γ. However the two parameterizations change the order in which the points occur. In the first parameterization, the point $z = 3/4$ occurs "after" the point $z = 1/2$, whereas in the second parameterization $z = 3/4$ occurs "before" (at $t = 1/4$) $z = 1/2$ (which occurs at $t = 1/2$).

Similarly $z_1(t) = \cos t + i\sin t$, $0 \le t \le \pi/2$ and $z_2(t) = \sin t + i\cos t$, $0 \le t \le \pi/2$ parameterize the arc of the unit circle (centered at 0) which lies in the first quadrant and (on the positive axes). However the order in which the points occur is reversed in the two parameterizations.



There are only two "natural" orderings of any smooth arc and this can be specified by specifying the initial point.

Definition: A directed smooth arc is a smooth arc with a specific ordering of its points.

A directed smooth closed curve is a little more complicated to define since the initial point is the terminal point too. However we can specify an ordering of the remaining points. This amounts to specifying the "direction of transit" from initial point. We say the points of a smooth closed curve have been ordered if (i) we select an initial point, and (ii) select a "direction of transit". A smooth closed curve whose points have been ordered is called a directed smooth closed curve.

A directed smooth curve is either a directed smooth arc or a directed smooth closed curve.

Parameterizations reflect the ordering and hence the direction of the smooth curve.

Contours: A contour Γ is either a single point z_0 or a finite sequence of directed smooth curves $(\gamma_1, \gamma_2, \ldots, \gamma_n)$ such that the terminal point of γ_k coincides with the initial point of γ_{k+1}, $k = 1, \ldots, n-1$. We write

$$\Gamma = \gamma_1 + \ldots + \gamma_n.$$

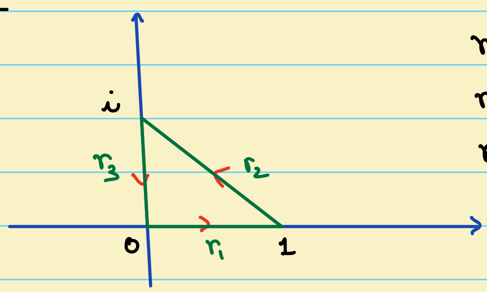
One can piece together the admissible parametrizations of the γ_i to get a parametrize contours.

If $\Gamma = \gamma_1 + \ldots + \gamma_n$, one can find an interval $[a, b]$ and points $\tau_0, \tau_1, \ldots, \tau_n$ in $[a, b]$ s.t.

$$a = \tau_0 < \tau_1 < \ldots < \tau_{n-1} < \tau_n = b$$

and a parametrization $z(t)$, $a \le t \le b$ of Γ such that on the subinterval $[\tau_{k-1}, \tau_k]$, $z(t)$ is an admissible parametrization of γ_k.

Example:



γ_1 : $z(t) = t$ $\quad 0 \le t \le 1$

γ_2 : $z(t) = 1 + t(i-1)$, $\quad 0 \le t \le 1$

γ_3 : $z(t) = i - ti$, $\quad 0 \le t \le 1$.

Putting it together:

$$z(t) = \begin{cases} t & 0 \le t \le 1 \\ 1 + (t-1)(i-1), & 1 \le t \le 2 \\ i - (t-2)i, & 2 \le t \le 3. \end{cases}$$

So have $z(t)$, $0 \le t \le 3$.

The book scales the above down to $[0, 1/3]$, $[1/3, 2/3]$, $[2/3, 1]$, but I see no advantage in that.

Definition: Let γ be a smooth directed curve and f a continuous complex valued function whose domain contains γ.
Then the contour integral $\int_\gamma f(z)dz$ of f over γ is

$$\int_\gamma f(z)\,dz := \int_a^b f(z(t))\,z'(t)\,dt$$

where $z(t)$, $a \le t \le b$ is an admissible parametrization of γ.

In practical terms: if $f = u + iv$ and $z(t) = x(t) + iy(t)$ then

$$\int_\gamma f(z)dz = \int_a^b \{u(x(t), y(t)) + i v(x(t), y(t))\}(x'(t) + i y'(t))\,dt$$

$$= \int_a^b \{u(x(t), y(t))\,x'(t) - v(x(t), y(t))\,y'(t)\}\,dt$$

$$+\ i\int_a^b \{v(x(t), y(t))\,x'(t) + u(x(t), y(t))\,y'(t)\}\,dt$$

$$= \underbrace{\int_\gamma (u\,dx - v\,dy)} + i \underbrace{\int_\gamma (v\,dx + u\,dy)}$$

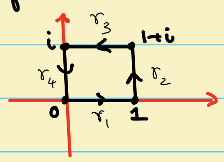
line integrals

Since we have written this as a line integral it is independent of parametrization.

Definition: If $\Gamma = \gamma_1 + \gamma_2 + \dots + \gamma_n$ is a contour then the contour integral of a complex valued function f defined in a neighbourhood of Γ is

$$\int_\Gamma f(z)\,dz = \sum_{i=1}^{n} \int_{\gamma_i} f(z)\,dz.$$

Example: Find $\int_\Gamma \bar{z}\, dz$ where Γ is the contour given by unit square oriented in the counter-clockwise direction. Here the unit square means the square with vertices 0, 1, $1+i$, and i.



$$\Gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$$

Solution:

$\gamma_1 : \quad z(t) = t \qquad 0 \le t \le 1$

$\gamma_2 : \quad z(t) = 1 + it \qquad 0 \le t \le 1$

$\gamma_3 : \quad z(t) = 1 - t + i \qquad 0 \le t \le 1$

$\gamma_4 : \quad z(t) = (1-t)\, i \qquad 0 \le t \le 1.$

$\bar{z}(t) = x(t) - i y(t).$

(i) $\displaystyle\int_{\gamma_1} \bar{z}\, dz = \int_0^1 (t - i.0)\, dt = \int_0^1 t\, dt = \frac{1}{2}$

(ii) $\displaystyle\int_{\gamma_2} \bar{z}\, dz = \int_0^1 (1-it)(i\, dt) = \frac{1}{2} + it.$

(iii) $\displaystyle\int_{\gamma_3} \bar{z}\, dz = \int_0^1 \{(1-t) - i\}(-dt) = -\frac{1}{2} + it$

(iv) $\displaystyle\int_{\gamma_4} \bar{z}\, dz = \int_0^1 \{-(1-t)i\}(-i\, dt) = -\int_0^1 (1-t)\, dt = -\frac{1}{2}.$

Thus $\displaystyle\int_\Gamma \bar{z}\, dz = \frac{1}{2} + \left(\frac{1}{2} + it\right) + \left(-\frac{1}{2} + it\right) + \left(-\frac{1}{2}\right) = 2i.$

Chain Rule:

Suppose γ is a smooth directed curve, f a function in a domain D containing γ and F an analytic function on D s.t.

$$F'(z) = f(z).$$

Let the initial point of γ be z_0 and the terminal point z_1. Suppose we have an admissible parametrization of γ

$$\gamma : \quad z(t), \quad a \le t \le b.$$

Then it is easy to see that

$$\frac{d}{dt} F(z(t)) = f(z(t))\, z'(t).$$

This gives:

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)\,dt = \int_a^b \frac{d}{dt} F(z(t))\,dt$$
$$= F(z(b)) - F(z(a))$$
$$= F(z_1) - F(z_0).$$

So in this case $\int_\gamma f(z)dz$ depends only on the end points of γ. In particular if γ is smooth directed <u>closed</u> curve, then
$$\int_\gamma f(z)\,dz = 0.$$

If $\Gamma = \gamma_1 + \dots + \gamma_n$ is a Contour, then (in the above case), if z_0 is the initial point of Γ (and hence of γ_1), and $z_1, \dots, z_n$ the terminal points of $\gamma_1, \dots, \gamma_n$ respectively, then

$$\begin{aligned}\int_\Gamma f(z)\,dz = \; & F(\cancel{z_1}) - F(z_0) \\ & + F(\cancel{z_2}) - F(\cancel{z_1}) \\ & \vdots \\ & + F(\cancel{z_{n-1}}) - F(\cancel{z_{n-2}}) \\ & + F(z_n) - F(\cancel{z_{n-1}}) \\ \\ = \; & F(z_n) - F(z_0).\end{aligned}$$

Once again, if Γ is closed, this means
$$\int_\Gamma f(z)dz = 0.$$

Theorem: Suppose Γ is a contour in a domain D with initial point w and terminal w^*, f a function on D which has an anti-derivative say F (i.e. $F'(z) = f(z)$ on D) then
$$\int_\Gamma f(z)\,dz = F(w^*) - F(w).$$

Example: We have already seen that if $n \neq -1$, $\int_C z^n dz = 0$ where C is the circle $|z| = 2$ oriented in the counter clockwise direction. This can be also proved by noting that if $n \neq -1$, $\dfrac{z^{n+1}}{n+1}$ is an anti-derivative

on z^n in a domain D containing C. If $n \leq -2$, one can pick D to be $\mathbb{C} \setminus \{0\}$ and if $n \geq 0$, $D = \mathbb{C}$ works.

Reversing the direction : Suppose γ is a smooth directed curve. Let $-\gamma$ be the directed curve whose ordering of points is the reverse of the ordering for γ. If

$$\gamma : \quad z(t), \quad a \leq t \leq b$$

is an admissible parametrization of γ then

$$-\gamma : \quad z(-t), \quad -b \leq t \leq -a$$

is an admissible parametrization of $-\gamma$.

It follows that

$$\int_{-\gamma} f(z)\,dz = \int_{-b}^{-a} f(z(-t))\,(-z'(-t))\,dt$$

$$= \int_{b}^{a} f(z(s))\,(-z'(s))\,(-ds)$$

$$= \int_{b}^{a} f(z(s))\,z'(s)\,ds$$

$$= -\int_{a}^{b} f(z(s))\,z'(s)\,ds$$

$$= -\int_{\gamma} f(z)\,dz.$$

One can similarly talk about $-\Gamma$ for a contour $\Gamma = (\gamma_1, \dots, \gamma_n)$.
$-\Gamma = (-\gamma_n, -\gamma_{n-1}, \dots, -\gamma_2, -\gamma_1)$, and it is clear that

$$\boxed{\int_{-\Gamma} f(z)\,dz = -\int_{\Gamma} f(z)\,dz.}$$