Poles and zeros of rational functions

Let
$$R(z) = c \cdot \frac{(z-\zeta_1)^{e_1} \cdots (z-\zeta_m)^{e_m}}{(z-z_1)^{d_1} \cdots (z-z_k)^{d_k}}$$
be a rational function with c a non-zero constant, and $\zeta_1, \dots, \zeta_m, z_1, \dots, z_k$ distinct complex numbers.

- $\zeta_1, \dots, \zeta_m$ are called the zeros of $R(z)$, with ζ_j being a zero of order e_j.
- $z_1, \dots, z_k$ are called the poles of $R(z)$, with z_i being a pole of order d_i.

Taylor form centred at z_0

If
$$p(z) = a_0 + a_1 z + \dots + a_n z^n$$
is a polynomial of degree n, and z_0 is a complex number, we saw that we can re-write $p(z)$ as
$$p(z) = b_0 + b_1(z-z_0) + b_2(z-z_0)^2 + \dots + b_n(z-z_0)^n \quad \text{—}(*)$$
and that $b_0, \dots, b_n$ are given by the formula
$$b_k = \frac{p^{(k)}(z_0)}{k!}$$
where $p^{(k)}(z)$ denotes the k^{th} derivative of p.

The form $(*)$ is called the Taylor form of p centred at z_0. It is also called the Taylor's expansion of p around z_0.

Example: Find the Taylor form of $p(z) = z^5$ centered at 3.

Soln:

Method I: $p(z) = z^5$, $p'(z) = 5z^4$, $p''(z) = 20z^3$, $p^{(3)}(z) = 60z^2$, $p^{(4)}(z) = 120z$, $p^{(5)}(z) = 120$.

Now $b_k = p^{(k)}(3)/k!$, and hence

$b_0 = 3^5$, $b_1 = 5(3^4)/1!$, $b_2 = 20(3^3)/2!$, $b_3 = 60(3^2)/3!$

$b_4 = 120(3)/4!$, $b_5 = \frac{120}{5!}$.

Thus $b_0 = 243$, $b_1 = 405$, $b_2 = 270$, $b_3 = 90$, $b_4 = 15$, $b_5 = 1$.
In other words

$$z^5 = \underbrace{243 + 405(z-3) + 270(z-3)^2 + 90(z-3)^3 + 15(z-3)^4 + (z-3)^5}_{\text{Taylor's form of } z^5 \text{ centered at } 3.}.$$

Method II : Use the binomial theorem.

$$z^5 = [(z-3)+3]^5$$
$$= \sum_{k=0}^{5} \binom{5}{k} (3)^{5-k} (z-3)^k.$$

You will get the same answer.
Either method is fine.

Arg, arg, Log, log, multiple-valued functions, branches

Recall that arg is a multiple-valued function on $\mathbb{C} \setminus \{0\}$. So is log.

$$\log z = \underbrace{\log|z|}_{\text{single-value}} + i\underbrace{\arg(z)}_{\text{multiple-valued}} = \underbrace{\log r}_{\text{single-valued}} + i\underbrace{\theta}_{\text{multiple-valued}}.$$

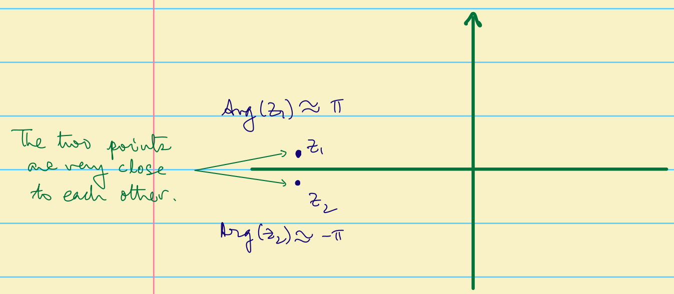
Let $\tau \in \mathbb{R}$. Then $\arg_\tau(z)$ is, by definition, the unique value of $\arg(z)$ lying in the interval $(\tau, \tau+2\pi]$. The function $\arg_\tau$ is single-valued.

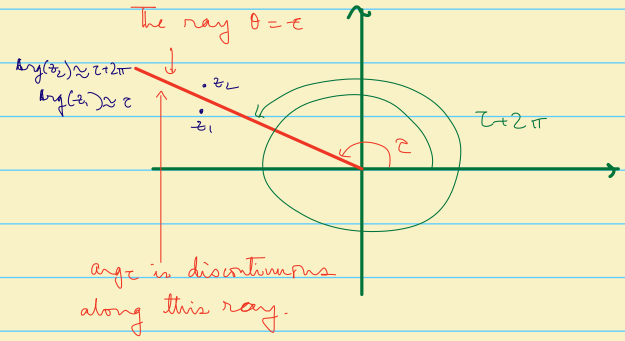
"Formulas"

(i) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
(ii) $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$
(iii) $\log(z_1 z_2) = \log(z_1) + \log(z_2)$
(iv) $\log\left(\frac{z_1}{z_2}\right) = \log(z_1) - \log(z_2)$

Interpretation: If particular values are assigned to any two of the terms then one can find a value of the third term so that the equation is true.

Branch cut for Arg, Log, $\arg_\tau$, $\mathcal{L}_\tau$: Recall that Arg is NOT continuous on the negative real axis (or at 0). Two points z_1 and z_2 close to each other, with z_1 in the 2nd quadrant and z_2 in the 3rd, will have Arg values which are not close: $\mathrm{Arg}(z_1) \approx \pi$, but $\mathrm{Arg}(z_2) \approx -\pi$. This is why Arg is not continuous on the negative real axis.





The "cut" along $(-\infty, 0]$ for Arg and Log is called a <u>branch cut</u>. We point out that $\mathrm{Arg} = \arg_{-\pi}$, and $\arg_\tau$ is discontinuous along the ray $\theta = \tau$ (we include 0 in the ray). Call this ray R_τ. The R_τ is called the branch cut for $\arg_\tau$.

There is then a version of logarithm for each τ, namely
$$\mathcal{L}_\tau(z) = \log|z| + i\arg_\tau(z).$$
The Ray R_τ is also a branch cut for the above version of a logarithm, i.e. of $\mathcal{L}_\tau$. Note $\mathcal{L}_{-\pi} = \mathrm{Log}$. These different versions of the logarithm are called <u>branches of the multiple-valued function</u> <u>$\log(z)$</u>.

$\mathcal{L}_\tau(z)$ is analytic outside its branch cut and is discontinuous on its branch cut R_τ. Moreover on $\mathbb{C} - R_\tau$ we have
$$\frac{d}{dz}\mathcal{L}_\tau(z) = \frac{1}{z}.$$

The proof is the same as the one we gave for the $\frac{d}{dz}\mathrm{Log}(z) = \frac{1}{z}$. In this case just take $\theta = \arg_\tau(z)$.

<u>Definition</u>: $F(z)$ is said to be a <u>branch</u> of a multiple-valued function $f(z)$ in a domain D if $F(z)$ is single-valued on D and has the property that for each $z \in D$, the value $F(z)$ is one of the values of $f(z)$.

For example, $\arg_\tau$ is a branch of arg, and $\mathcal{L}_\tau$ is a branch of log.

<u>Examples</u>:

1. Is $|\cos z|$ bounded?

<u>Ans</u>: No. Here are the computations:
$$\cos(iy) = \frac{e^{i(iy)} + e^{-i(iy)}}{2} = \frac{e^{-y} + e^{y}}{2} \to \infty \text{ as } y \to \infty.$$

It follows that $\cos(z)$ is not bounded.

2. By the chain rule, if f and g are analytic and the range of g is contained in the domain of f, then $f \circ g$ is analytic.
As an example
$$f(z) = \cos(z^3) - e^{-7z} + iz^2$$
is entire.

Complex Powers

Let α and z be complex numbers with $z \neq 0$. Define z^α as the multi-valued expression

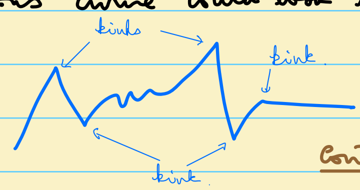
$$\boxed{z^\alpha = e^{\alpha \log z}}$$

<u>Question</u>: With the above definition, are all powers of 1 equal to ?

Contour Integration

Here is an <u>informal</u> introduction. A more formal introduction will happen in the next lecture.

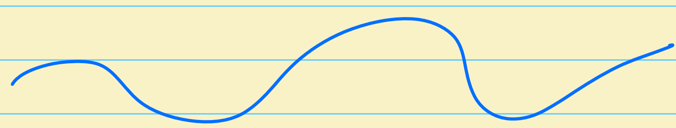
A continuous curve could look like this:



Continuous curve

The pen hasn't left the paper (or, the electronic pencil has not left the tablet) while drawing the curve, making it "continuous". However, there may well be sharp corners, i.e. kinks.

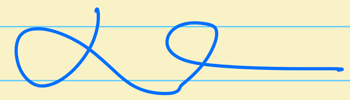
A smooth curve will not have these kinks.



Smooth curve

A smooth curve is continuous.

We would also like our curves to not cross themselves like this

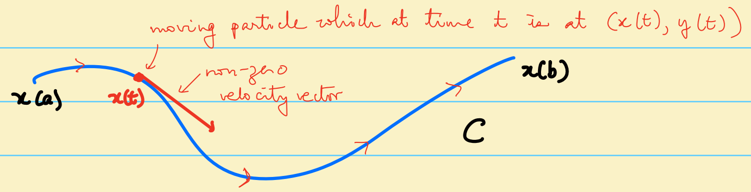


One achieves this by parametrization, i.e. if a point on the curve is a function of time $x = x(t)$, $y = y(t)$, $a \le t \le b$, with the requirement that $(x(t), y(t)) \neq (x(s), y(s))$ if $t \neq s$, $a \le t, s \le b$.

The requirement of continuity is met by requiring that $x(t)$ and $y(t)$ are continuous functions of t.

The smoothness requirement is met by requiring that $x(t)$ and $y(t)$ be differentiable, i.e. $x'(t)$ and $y'(t)$ exist on $[a,b]$ (at the end point $t=a$ we want the right derivative to exist and at $t=b$, we want the left derivative to exist).

We would also like our parametrization to be such that the "velocity vector" at every point is non-zero, i.e., $(x'(t), y'(t)) \neq (0,0)$ for any $a \le t \le b$.



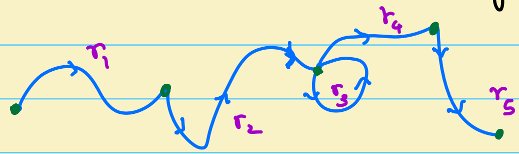
A curve C with these properties is called a smooth arc (we will define this term more formally next time). But to summarize, C is a smooth arc if:

- C can be parametrized: $x = x(t)$, $y = y(t)$, $a \le t \le b$.
- $x(t)$ and $y(t)$ are differentiable on $[a,b]$ (at the end points $t=a$ and $t=b$, the derivatives are the appropriate one-sided derivatives.
- $(x'(t), y'(t)) \neq (0,0)$ for $a \le t \le b$
- If $t \neq s$, $a \le t, s \le b$, then $(x(t), y(t)) \neq (x(s), y(s))$.

A smooth closed curve C has all the properties above, except we require $(x(a), y(a)) = (x(b), y(b))$. This is the one exception to the last rule.

We prefer writing
$$z(t) = x(t) + i\,y(t). \qquad a \le t \le b$$
for the above parametrization.

There is one more concept: that of a contour. Roughly a contour Γ is a finite number of smooth arcs or smooth loops $\gamma_1, \dots, \gamma_n$, with the initial point of γ_{i+1} being the end point of γ_i. We also allow single points to be contours in addition to the kind just described.



Note that there is a direction that a moving point on a contour is traversing.

NOTE the upgraded definition, which is little more precise than the earlier one.

By a smooth curve we mean either a smooth arc or a smooth loop. Thus a contour is a sequence of smooth arcs, the end point of one curve being the initial point of the next curve.

So suppose C is a smooth curve, and say

$$z(t) = x(t) + iy(t), \quad a \le t \le b$$

is an admissible parameterization of C (i.e. a parameterization which follows all the requirements we put in).

Write $z'(t)$ for $x'(t) + iy'(t)$, $a \le t \le b$.

$$z'(t) := x'(t) + iy'(t).$$

Let f be a continuous function on a set which contains C.

Define

$$\int_C f(z)\,dz := \int_a^b f(z(t))\, z'(t)\,dt.$$

It is easy to see that $\int_C f(z)dz$ does not depend on the way it is parameterized.

If $\Gamma = \gamma_1 + \dots + \gamma_n$ is a contour with γ_i smooth curves then define

$$\int_\Gamma f(z)\,dz = \sum_{i=1}^{n} \int_{\gamma_i} f(z)\,dz$$

This has been defined above.

We will do an example in the next page.

Example: Let C be the circle $|z|=2$ traversed once in the counter-clockwise direction. Compute

$$\int_C z^n \, dz.$$

Solution:

Parametrization:

$$z(t) = 2(\cos(t) + i\sin(t)) = 2e^{it}, \qquad 0 \le t \le 2\pi$$

$$\int_C z^n dz = \int_0^{2\pi} (2e^{it})^n (2ie^{it}) \, dt$$

$$= 2^{n+1} i \int_0^{2\pi} e^{i(n+1)t} \, dt.$$

Case 1: $n=-1$. Then $e^{i(n+1)t} = 1$ and $2^{n+1} = 1$

$$\int_C z^n dz = i\int_0^{2\pi} dt = 2\pi i.$$

Case 2: $n \neq -1$. Then

$$\int_C z^n dz = 2^{n+1} i \int_0^{2\pi} \{\cos((n+1)t) + i\sin((n+1)t)\} dt$$

$$= 2^{n+1} i \left\{ \int_0^{2\pi} \cos((n+1)t)dt + i\int_0^{2\pi} \sin((n+1)t)\, dt \right\}$$

Now $\displaystyle\int_0^{2\pi} \cos((n+1)t)dt = \frac{1}{n+1}\Big[\sin((n+1)t)\Big]_0^{2\pi}$

$$= \frac{1}{n+1}\Big[\sin((n+1)(2\pi)) - \sin(0)\Big]$$

$$= 0.$$

Similarly $\displaystyle\int_0^{2\pi} \sin((n+1)t)\,dt = -\left[\frac{\cos((n+1)t)}{n+1}\right]_0^{2\pi} = 0.$

Thus

$$\int_C z^n dz = \begin{cases} 2\pi i & \text{if } n=-1 \\ 0 & \text{otherwise.} \end{cases}$$

If $n \neq -1$, z^n has an anti-derivative, namely $\frac{z^n}{n+1}$. We'll see later that this is another reason $\int_C z^n dz = 0$ for $n \neq -1$.