Lecture 10

Polar Cauchy-Riemann equations

If $f=u+iv$ is a function on an open set G in $\mathbb{C}$ with $0 \notin G$, then writing $f(r,\theta)$, $u(r,\theta)$, $v(r,\theta)$ for $f(re^{i\theta})$, $u(re^{i\theta})$, $v(re^{i\theta})$, we can regard f, u, and v as functions of (r,θ). The polar Cauchy-Riemann equations for f (or for u and v) are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

The usual Cauchy-Riemann equations, i.e. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, are often called the rectangular Cauchy-Riemann equations when we wish to distinguish it from the polar form of the equations. The two forms are equivalent as the following as the following result shows.

Theorem: The function f satisfies the polar Cauchy-Riemann equations on G if and only if it satisfies the rectangular Cauchy-Riemann equations on G.

Proof:

Consider the matrix

$$A_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \quad \longleftarrow \text{represents rotation by an angle of } -\theta.$$

R_θ gives us rotation by an angle of $-\theta$. Let

$$T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad \longleftarrow \text{represents rotation by } -\frac{\pi}{2}$$

Then T represents rotation by $-\pi/2$.

It follows that

$$TA_\theta = A_\theta T \quad \cdots\cdots \text{(A)}$$

since both sides give rotation by $-(\pi/2+\theta)$.

A_θ appears in the following calculation in multivariable calculus, when we compare cartesian (rectangular) coordinates with polar coordinates. Let h be a function on $G \subset \mathbb{C}-\{0\}$ such that h has both partial derivatives.

Now $x = r\cos\theta$, $y = r\sin\theta$.

The chain rule in multivariable calculus gives

$$\partial h/\partial r = \frac{\partial x}{\partial r}\frac{\partial h}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial h}{\partial y} \quad \text{and} \quad \frac{\partial h}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial h}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial h}{\partial y}.$$

Thus $\frac{\partial h}{\partial r} = \cos\theta \frac{\partial h}{\partial x} + \sin\theta \frac{\partial h}{\partial y}$ and $\frac{\partial h}{\partial \theta} = -r\sin\theta \frac{\partial h}{\partial x} + \cos\theta \frac{\partial h}{\partial y}$.

This can be re-written as

$$\begin{pmatrix} \frac{\partial h}{\partial r} \\ \frac{1}{r}\frac{\partial h}{\partial \theta} \end{pmatrix} = A_\theta \begin{pmatrix} \frac{\partial h}{\partial x} \\ \frac{\partial h}{\partial y} \end{pmatrix} \quad \text{———} \quad ①$$

Multiply both sides by T and use the formula $TA_\theta = A_\theta T$ to get $T\begin{pmatrix} \partial h/\partial r \\ \frac{1}{r}\partial h/\partial\theta \end{pmatrix} = A_\theta T \begin{pmatrix} \partial h/\partial x \\ \partial h/\partial y \end{pmatrix}$. Since $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ -a \end{pmatrix}$, this gives

$$\begin{pmatrix} \frac{1}{r}\frac{\partial h}{\partial \theta} \\ -\frac{\partial h}{\partial r} \end{pmatrix} = A_\theta \begin{pmatrix} \partial h/\partial y \\ -\partial h/\partial x \end{pmatrix} \quad \text{———} \quad ②$$

Apply ① to $h = u$ and ② to $h = v$ to get

$$\begin{pmatrix} \frac{\partial u}{\partial r} \\ \frac{1}{r}\frac{\partial u}{\partial \theta} \end{pmatrix} = A_\theta \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} \quad \text{———} \quad Ⓐ$$

and

$$\begin{pmatrix} \frac{1}{r}\frac{\partial v}{\partial \theta} \\ -\frac{\partial v}{\partial r} \end{pmatrix} = A_\theta \begin{pmatrix} \frac{\partial v}{\partial y} \\ -\frac{\partial v}{\partial x} \end{pmatrix} \quad \text{———} \quad Ⓑ$$

From Ⓐ and Ⓑ we see that

$$\begin{pmatrix} \frac{\partial u}{\partial r} \\ \frac{1}{r}\frac{\partial u}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \frac{1}{r}\frac{\partial v}{\partial \theta} \\ -\frac{\partial v}{\partial r} \end{pmatrix} \text{ if and only if } A_\theta \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = A_\theta \begin{pmatrix} \frac{\partial v}{\partial y} \\ -\frac{\partial v}{\partial x} \end{pmatrix} \underbrace{\text{ if and only if }}_{\substack{\text{since } A_\theta \text{ is invertible} \\ (\text{note } \det A_\theta = \cos^2\theta + \sin^2\theta = 1 \neq 0)}} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial v}{\partial y} \\ -\frac{\partial v}{\partial x} \end{pmatrix}.$$

In other words

Polar C-R hold for $f = u + iv$ if and only if rectangular C-R hold for $f = u + iv$. //

Remark: There is a simpler geometric reason why A_θ is invertible. Since it gives rotation by $-\theta$ radians, it has an inverse, namely rotation by $+\theta$ radians, i.e. $A_\theta^{-1} = A_{-\theta}$, i.e.,

$$A_\theta^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad (\text{for } \cos(-\theta) = \cos(\theta) \text{ \& } \sin(-\theta) = -\sin\theta).$$

It follows from our calculations earlier that

$$\begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = A_\theta^{-1} \begin{pmatrix} \frac{\partial u}{\partial r} \\ \frac{1}{r}\frac{\partial u}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial r} \\ \frac{1}{r}\frac{\partial u}{\partial \theta} \end{pmatrix}$$

Now suppose $f = \text{Log}$, the principal logarithm, and $G = D = \mathbb{C} \setminus (-\infty, 0]$. Then, as we saw in the last class, if $f = u + iv$, then

$$u = \log r \quad \text{and} \quad v = \theta$$

giving us
$$\begin{cases} \frac{\partial u}{\partial r} = \frac{1}{r}, & \frac{\partial v}{\partial r} = 0 \\ \frac{\partial u}{\partial \theta} = 0, & \frac{\partial v}{\partial \theta} = 1 \end{cases}$$

Thus Log satisfies the polar C-R eqns, whence the usual C-R eqns. It follows that Log is analytic.

Moreover since
$$\begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial r} \\ \frac{1}{r}\frac{\partial u}{\partial \theta} \end{pmatrix}$$

we get
$$\begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{1}{r} \\ 0 \end{pmatrix}$$

whence
$$\frac{\partial u}{\partial x} = \frac{1}{r}\cos\theta, \quad \frac{\partial u}{\partial y} = \frac{1}{r}\sin\theta.$$

Now
$$\begin{aligned} \frac{d}{dz}\text{Log}(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \text{(by C-R eqns)} \\ &= \frac{1}{r}\cos\theta - \frac{1}{r}\sin\theta \\ &= \frac{1}{r}e^{-i\theta} \\ &= \frac{1}{re^{i\theta}} \\ &= \frac{1}{z}. \end{aligned}$$

We have therefore proven:

Theorem: Log is analytic on $D = \mathbb{C} \setminus (-\infty, 0]$ and
$$\frac{d}{dz}\left(\operatorname{Log}(z)\right) = \frac{1}{z}.$$

Example: Find all solutions of $e^z = -1$.

Solution:

Let $z = x + iy$. The eqn $e^z = -1$ translates to
$$e^x e^{iy} = -1.$$
Thus $e^x = |e^x e^{iy}| = |-1| = 1$. Since $x \in \mathbb{R}$ the eqn $e^x = 1$ has only one solution, namely $x = 0$. Thus
$$z = iy.$$
We have to solve
$$e^{iy} = -1 \qquad \text{where } y \in \mathbb{R}.$$
So
$$e^{iy} = e^{i\pi}$$
$$\Rightarrow \quad e^{i(y-\pi)} = 1.$$
If $\theta \in \mathbb{R}$ and $e^{i\theta} = 1$, then $\cos\theta + i\sin\theta = 1$, which means:
$$\cos\theta = 1 \quad \text{and} \quad \sin\theta = 0.$$
The only solutions of this are
$$\theta = 2\pi n, \quad n \in \mathbb{Z}.$$
Since $e^{i(y-\pi)} = 1$, it follows that
$$y - \pi = 2\pi n, \quad n \in \mathbb{Z}.$$
Thus $y = (2n+1)\pi, \quad n \in \mathbb{Z}.$

Now $z = x + iy = 0 + iy = iy.$

Hence the solutions of the eqn $e^z = -1$ are
$$z = (2n+1)i\pi, \quad n \in \mathbb{Z}.$$
odd multiples of $i\pi$.

Other Entire functions:

If x is real, it is easy to see that
$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

We extend this definition to $\mathbb{C}$ in the obvious way.

$$\cos z := \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z := \frac{e^{iz} - e^{-iz}}{2i}, \quad z \in \mathbb{C}.$$

We can also extend the definition of hyperbolic cosine and sine from $\mathbb{R}$ to $\mathbb{C}$ in the obvious manner, namely

$$\underbrace{\cosh(z)}_{\text{The hyperbolic cosine}} = \frac{e^z + e^{-z}}{2}, \quad \underbrace{\sinh(z)}_{\text{The hyperbolic sine}} = \frac{e^z - e^{-z}}{2}$$

Since e^z is an entire function, it is clear that $\cos(z)$, $\sin(z)$, $\cosh(z)$ and $\sinh(z)$ are entire.

Theorem: The functions $\cos(z)$, $\sin(z)$, $\cosh(z)$, $\sinh(z)$ are entire. Moreover the following identities hold.

(a) $\cos(-z) = \cos(z)$, $\sin(-z) = -\sin(z)$, $z \in \mathbb{C}$

(b) $\cosh(-z) = \cosh(z)$, $\sinh(-z) = -\sinh(z)$, $z \in \mathbb{C}$

(c) $\cos^2(z) + \sin^2(z) = 1$, $z \in \mathbb{C}$

(d) $\cosh^2(z) - \sinh^2(z) = 1$, $z \in \mathbb{C}$

(e) $\cos(z_1 + z_2) = \cos(z_1)\cos(z_2) - \sin(z_1)\sin(z_2)$, $z_1, z_2 \in \mathbb{C}$

(f) $\sin(z_1 + z_2) = \sin(z_1)\cos(z_2) + \cos(z_1)\sin(z_2)$, $z_1, z_2 \in \mathbb{C}$

Proof:

These are easy computations which are left to you. We give two examples of the computations:

$$\cos^2 z + \sin^2 z = \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2$$

$$= \frac{(e^{iz} + e^{-iz})^2}{4} - \frac{(e^{iz} - e^{-iz})^2}{4} \qquad \text{(using the fact that } i^2 = -1\text{)}$$

$$= \frac{\cancel{e^{2iz}} + 2 + \cancel{e^{-2iz}} - (\cancel{e^{2iz}} - 2 + \cancel{e^{-2iz}})}{4}$$

$$= \frac{4}{4}$$

$$= 1.$$

Also

$$\cosh^2(z) - \sinh^2(z) = \frac{(e^z + e^{-z})^2 - (e^z - e^{-z})^2}{4} = \frac{\cancel{e^{2z}} + 2 + \cancel{e^{-2z}} - \cancel{e^{2z}} + 2 - \cancel{e^{-2z}}}{4} = \frac{4}{4} = 1.$$

The remaining calculation are left for you. //

Differentiation of these functions:

Theorem: (a) $\frac{d}{dz}\cos(z) = -\sin z$

(b) $\frac{d}{dz}\sin(z) = \cos z$

(c) $\frac{d}{dz}\cosh(z) = \sinh(z)$

(d) $\frac{d}{dz}\sinh(z) = \cosh(z)$

Proof:

All the assertions follow from $\frac{d}{dz}e^z = e^z$ and the chain rule. The details are left to you. //

Other trig functions:

$$\left.\begin{aligned} \sec(z) &:= \frac{1}{\cos z} \\ \tan(z) &:= \frac{\sin(z)}{\cos(z)} \end{aligned}\right\} \text{ Defined where } \cos(z) \neq 0$$

$$\left.\begin{aligned} \operatorname{cosec}(z) &:= \frac{1}{\sin(z)} \\ \cot(z) &:= \frac{\cos(z)}{\sin(z)} \end{aligned}\right\} \text{ Defined where } \sin(z) \neq 0.$$